On the chordality of polynomial sets in triangular decomposition in top-down style

Chenqi Mou

Abstract. In this talk, we show the connections between chordal graphs which permit perfect elimination orderings on their vertexes from Graph Theory and triangular decomposition which decompose polynomial sets into triangular sets from Computer Algebra and present the chordal graph structures of polynomial sets appearing in triangular decomposition in top-down style when the input polynomial set has a chordal associated graph. In particular, we show that the associated graph of one specific triangular set in any algorithm for triangular decomposition in top-down style is a subgraph of that chordal graph and that all the triangular sets computed by Wang's method for triangular decomposition have associated graphs which are subgraphs of that chordal graph. Furthermore, the associated graphs of polynomial sets can be used to describe their sparsity with respect to the variables, and we present a refined algorithm for efficient triangular decomposition for sparse polynomial sets in this sense.

This talk is based on the joint work with Yang Bai.

1. Chordal graphs and triangular decomposition

Let $\mathbb{K}$ be a field, and $\mathbb{K}[\boldsymbol{x}]$ be the multivariate polynomial ring over $\mathbb{K}$ in the variables $x_1, \ldots, x_n$.

For a polynomial $F \in \mathbb{K}[\boldsymbol{x}]$, define the (variable) *support* of F, denoted by $\operatorname{supp}(F)$, to be the set of variables in $x_1, \ldots, x_n$ which effectively appear in F. For a polynomial set $\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$, $\operatorname{supp}(\mathcal{F}) := \cup_{F \in \mathcal{F}} \operatorname{supp}(F)$, and its *associated graph* $G(\mathcal{F}) = (V, E)$ is an undirected graph with $V = \operatorname{supp}(\mathcal{F})$ and $E = \{(x_i, x_j) : \exists F \in \mathcal{F} \text{ such that } x_i, x_j \in \operatorname{supp}(F)\}$.

Let $G = (V, E)$ be a graph with $V = \{x_1, \ldots, x_n\}$. Then an ordering $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$ of the vertexes is called a *perfect elimination ordering* of G if for each $j = i_1, \ldots, i_n$, the restriction of G on the set $\{x_j\} \cup \{x_k : x_k < x_j \text{ and } (x_k, x_j) \in E\}$ is a clique. A graph G is said to be *chordal* if there exists a perfect elimination ordering of it and a polynomial set $\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$ is said to be *chordal* if $G(\mathcal{F})$ is chordal.

For example, the associated graph of $\mathcal{P} = \{x_2 + x_1, x_3 + x_1, x_4^2 + x_2, x_4^3 + x_3, x_5 + x_2, x_5 + x_3 + x_2\}$ is shown in Figure 1. One can find that the associated graph $G(\mathcal{P})$ is chordal by definition and thus $\mathcal{P}$ is chordal.

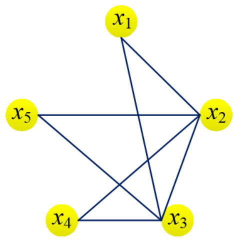

FIGURE 1. The associated graphs $G(\mathcal{P})$

Let the variables in $\mathbb{K}[\boldsymbol{x}]$ be ordered as $x_1 < \cdots < x_n$. An ordered set of non-constant polynomials $\mathcal{T} \subset \mathbb{K}[\boldsymbol{x}]$ is called a *triangular set* if the greatest variables of the polynomials in $\mathcal{T}$ increase strictly. A finite number of triangular sets $\mathcal{T}_1, \ldots, \mathcal{T}_r \subset \mathbb{K}[\boldsymbol{x}]$ are called a *triangular decomposition* of a polynomial set $\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$ if $\mathsf{Z}(\mathcal{F}) = \cup_{i=1}^{r}(\mathsf{Z}(\mathcal{T}_i) \setminus \mathsf{Z}(\prod_{T\in\mathcal{T}_i} \operatorname{ini}(T))$ holds, where $\operatorname{ini}(T)$ is the leading coefficient of T with respect to the greatest variable of T and $\mathsf{Z}(\cdot)$ denotes the set of common zeros.

Roughly speaking, an algorithm $\mathcal{A}$ for computing triangular decomposition of $\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$ is said to be in *top-down style* if the elimination of variables in $\mathcal{A}$ follows a strict order $x_n, x_{n-1}, \ldots, x_1$ and in the process of eliminating each x_i $(1 \leq i \leq n)$, no variables greater than x_i (namely $x_{i+1}, \ldots, x_n$) are generated.

2. Main theoretical results

2.1. General algorithms for triangular decomposition in top-down style

Denote the power set of a set S by 2^S. For an integer i $(1 \leq i \leq n)$, let f_i be a mapping

$$\begin{aligned} f_i : 2^{\mathbb{K}[\boldsymbol{x}_i]\setminus\mathbb{K}[\boldsymbol{x}_{i-1}]} &\to (\mathbb{K}[\boldsymbol{x}_i] \setminus \mathbb{K}[\boldsymbol{x}_{i-1}]) \times 2^{\mathbb{K}[\boldsymbol{x}_{i-1}]} \\ \mathcal{P} &\mapsto (T, \mathcal{R}) \end{aligned} \tag{1}$$

such that $\operatorname{supp}(T) \subset \operatorname{supp}(\mathcal{P})$ and $\operatorname{supp}(\mathcal{R}) \subset \operatorname{supp}(\mathcal{P})$. For a polynomial set $\mathcal{P} \subset \mathbb{K}[\boldsymbol{x}]$ and a fixed integer i $(1 \leq i \leq n)$, suppose that $(T_i, \mathcal{R}_i) = f_i(\mathcal{P}^{(i)})$ for some f_i as stated above. For $j = 1, \ldots, n$, define the polynomial set

$$\operatorname{red}_i(\mathcal{P}^{(j)}) := \begin{cases} \mathcal{P}^{(j)}, & \text{if } j > i \\ \{T_i\}, & \text{if } j = i \\ \mathcal{P}^{(j)} \cup \mathcal{R}_i^{(j)}, & \text{if } j < i \end{cases}$$

and $\mathrm{red}_i(\mathcal{P}) := \cup_{j=1}^{n} \mathrm{red}_i(\mathcal{P}^{(j)})$. In particular, write

$$\overline{\mathrm{red}}_i(\mathcal{P}) := \mathrm{red}_i(\mathrm{red}_{i+1}(\cdots(\mathrm{red}_n(\mathcal{P}))\cdots)) \tag{2}$$

for simplicity.

The mapping f_i in (1) is abstraction of specific reductions with respect to one variable x_i used in different kinds of algorithms for triangular decomposition in top-down style.

Theorem 2.1. *Let $\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$ be a chordal polynomial set with $x_1 < \cdots < x_n$ as one perfect elimination ordering and $\overline{\mathrm{red}}_i(\mathcal{F})$ be defined in* (2) *for $i = n, \ldots, 1$. Then the following statements hold:*

(a) *For each $i = n, \ldots, 1$, $G(\overline{\mathrm{red}}_i(\mathcal{F})) \subset G(\mathcal{F})$.*
(b) *If $\mathcal{T} := \overline{\mathrm{red}}_1(\mathcal{F})$ does not contain any nonzero constant, then $\mathcal{T}$ forms a triangular set such that $G(\mathcal{T}) \subset G(\mathcal{F})$.*

Theorem 2.1 (b) tells us that under the conditions stated in the theorem, the associated graph of one specific triangular set computed in any algorithm for triangular decomposition in top-down style is a subgraph of the associated graph of the input polynomial set. In fact, this triangular set is the "main branch" in the triangular decomposition in the sense that other branches are obtained by adding additional constrains in the splitting in the process of triangular decomposition.

2.2. Wang's method for triangular decomposition

A simply-structured algorithm was proposed by Wang for triangular decomposition in top-down style in 1993 [2]. The decomposition process in Wang's method applied to a polynomial set $\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$ can be viewed as a binary tree with its root as $(\mathcal{F}, \emptyset, n)$.

Theorem 2.2. *Let $\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$ be a chordal polynomial set with $x_1 < \cdots < x_n$ as one perfect elimination ordering and $(\mathcal{P}, \mathcal{Q}, i)$ be any node in the binary decomposition tree for Wang's method applied to $\mathcal{F}$. Then $G(\mathcal{P}) \subset G(\mathcal{F})$. In particular, let $\mathcal{T}_1, \ldots, \mathcal{T}_r$ be the triangular sets computed by Wang's method. Then $G(\mathcal{T}_i) \subset G(\mathcal{F})$ for $i = 1, \ldots, r$.*

As shown by Theorem 2.2, with a chordal input polynomial set, all the polynomials in the nodes of the decomposition tree of Wang's method, and thus all the computed triangular sets, have associated graphs which are subgraphs of that of the input polynomial set.

3. Fast triangular decomposition for variable sparse polynomial sets

Let $G(\mathcal{F}) = (V, E)$ be the associated graph of a polynomial set $\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$. Then the *variable sparsity* $s_v(\mathcal{F})$ of $\mathcal{F}$ can be defined as

$$s_v(\mathcal{F}) = |E| / \binom{2}{|V|},$$

where the denominator is the number of edges of a complete graph composed of $|V|$ vertexes. Triangular decomposition of a chordal and variable sparse polynomial set $\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$ with an algorithm in top-down style can be sped up by using the perfect elimination ordering of the chordal associated graph $G(\mathcal{F})$.

Some experimental comparisons of timings for computing regular decomposition of one class of chordal and variable sparse polynomials [1]

$$\mathcal{F}_i := \{x_k x_{k+3} - x_{k+1}x_{k+2} : k = 1, 2, \ldots, i\}, \quad i \in \mathbb{Z}_{>0}$$

with respect to the perfect elimination ordering versus random orderings are reported in the following table, where n denotes the variable number in $\mathcal{F}_i$, s_v denotes the variable sparsity, t_p and t_r are the timings (in seconds) for regular decomposition with respect to the perfect elimination orderings and 5 randomly chosen variable orderings respectively, and $\bar{t}_r$ are the average timings for random orderings.

Table 1. Timings for regular decomposition of $\mathcal{F}_i$

n	s_v	t_p	t_r					$\bar{t}_r$	$\bar{t}_r/t_p$
8	0.64	0.11	0.10	0.09	0.05	0.06	0.09	0.10	0.91
10	0.53	0.19	0.14	0.21	0.22	0.11	0.21	0.18	0.95
20	0.28	1.44	4.24	4.45	3.15	4.41	4.65	4.18	2.90
25	0.23	4.25	50.62	20.29	15.55	25.01	35.10	29.31	6.90
30	0.19	11.94	177.37	185.94	130.54	142.97	103.42	148.05	12.40
35	0.17	42.33	560.56	291.35	633.43	320.98	938.45	548.95	12.97
40	0.15	161.11	1883.64	3618.04	4289.13	4013.99	2996.37	3360.23	20.86

References

[1] Diego Cifuentes and Pablo A Parrilo. Chordal networks of polynomial ideals. *SIAM J. Appl. Algebra Geom.*, 1(1):73–110, 2017.

[2] Dongming Wang. An elimination method for polynomial systems. *J. Symbolic Comput.*, 16(2):83–114, 1993.

Chenqi Mou
LMIB – School of Mathematics and Systems Science
Beijing Advanced Innovation Center for Big Data and Brain Computing
Beihang University, Beijing 100191, China
e-mail: chenqi.mou@buaa.edu.cn