Confluent Heun equation and equivalent first-order systems

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Introduction

Presented text is an enlargement and elaboration of other publication of the authors [1]. The new vector formulations of confluent Heun equation (further CHE) is proposed. In its turn the text specifies integral symmetries and relation to Painleve equations as obtained in [2].

Consider CHE with two Fuchsian singularities at finite points z_j, $j = 1, 2$ and an irregular singularity at infinity. It reads

$$L^1(D, z)w(z) = (\sigma(z)D^2 + \tau(z)D + (\omega(z) - th))w(z) = 0, \tag{1}$$

Here either

$$\begin{aligned} \sigma(z) &= z(z-1) \\ \tau(z) &= -z(z-1) + c(z-1) + dz \\ \omega(z) &= -az \end{aligned} \tag{2}$$

or

$$\begin{aligned} \sigma(z) &= z(z-t) \\ \tau(z) &= -z(z-t) + c(z-t) + dz \\ \omega(z) &= -az \end{aligned} \tag{3}$$

In both cases (2), (3) polynomials $\sigma(z)$ and $\tau(z)$ are of second degree in z. As the result, differential operator L^1 has dimension 1 according to [3]. Note that the chosen form of CHE (1), (2) corresponds to that in the book [4] however (1), (3) is different from it. The advantage of the latter presentaton is discussed in [1].

Parameter h is called the **accessory parameter**. The chosen factor in front of it, namely t, leads to to the following lemma.

Lemma 1. *Equation (1) with (3) is reduced to confluent hypergeometric equation at* $t = 0$.

Proof. Set $t = 0$ in (1) with (3). We obtain instead of (1) the confluent hypergeometric equation.

At choosing another factor not proportional t, the accessory parameter h is conserved in limiting equation.

The interest to CHE is growing in last decades [4, 5]. Firstly, it is more general comparative to confluent hypergeometric equation. Secondly, more and more physical applications arise.

Linear first order system

The confluent Heun equation can be linked to first-order linear systems. However, these links can be different. One possible way of choosing such a system is determined by the demand that the the residues at Fuchsian points have zero determinant. In the other approach traces of these residues are taken to be zero. We study the first case here. Let the first-order system be

$$\vec{Y'(z)} = A(z)\vec{Y(z)}, \quad T(z) = \sigma(z)A(z) \tag{4}$$

where

$$A(z) = \frac{A^{(1)}}{z} + \frac{A^{(2)}}{z-t} + A^{(\infty)} \tag{5}$$

with

$$A^{(1)} = \begin{pmatrix} 0 & 0 \\ h & \theta_1 \end{pmatrix} \quad A^{(2)} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & \theta_2 - a_{11} \end{pmatrix} \quad A^{(\infty)} = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$$

The condition

$$\det A^{(2)} = 0$$

implies

$$a_{11}(\Theta_2 - a_{11}) - a_{21}a_{12} = 0$$

Hence, we arrive to the following result for matrix T

$$T(z) = \begin{pmatrix} a_{11}z + tz(z-t) & a_{12}z \\ h(z-t) + a_{21}z & \theta_1(z-t) + (\theta_2 - a_{11})z \end{pmatrix} \tag{6}$$

Further computations give

$$\operatorname{tr} T = tz(z-t) + \theta_1(z-t) + \theta_2 z$$

$$\det T = \sigma(z)(a_{11}\theta_1 - a_{12}h - t^2\theta_1 + t(\theta_2 + \theta_1 - a_{11})z)$$

$$T_{12}\left(\frac{T_{11}}{T_{12}}\right)' = tz$$

In view of lemma 1 the matrix element a_{12} should be chosen as

$$a_{12} = t \tag{7}$$

The searched equation for the first component of vector $\vec{Y(z)}$ reads

$$\sigma(z)y_1''(z) + P(z)y_1'(z) + Q(z)y_1(z) = 0 \tag{8}$$

where

$$P(z) = -\sigma(z)\left(ln\frac{T_{12}}{\sigma}\right)' - trT = -tz(z-t) + (\theta_1+1)z + \theta_2(z-t)$$

$$Q(z) = T_{12}\left(\frac{T_{11}}{T_{12}}\right)' + \sigma(z)^{-1}detT =$$

$$zt(\theta_1+\theta_2-a_{11}+1) + a_{11}\theta_1 - th$$

The followinng relations between matrix elements and parameters of equation (1) hold

$$a = a_{11}-\theta_1-\theta_2, \quad c=\theta_2, \quad d=\theta_1+1 \tag{9}$$

Shift in accessory parameter is not essential.

Painlevé equation P^V

Painlevé equation is a nonlinear integrable equation, widely studied and applied in last decades. Recent researches in this field one can find, for instance, in collection of papers [8]. Our interests lay in bijection relation between Heun equations and Painlevé equations. [4, 7].

The approach presented in this paper serves as justification of heuristic antiquantization of Heun equation proposed in previous papers starting with publication in J. Phys. A.: Math. Gen. [9].

We shortly repeat derivation of P^V. The transformation of a Hamiltonian to a Lagrangian consist of transfer from variable μ to variable $\dot{q}$ and transfer from Hamiltonian $H(\mu, q)$ to Lagrangian $\mathcal{L}(\dot{q}, q)$ according to

$$\dot{q} = \frac{\partial H}{\partial \mu} = \frac{2\sigma(q)\mu + \tau(q)}{t}$$

$$\mathcal{L}(\dot{q}, q) = \dot{q}\mu - H(\mu, q) =$$

$$\frac{((t)^{1/2}\dot{q} - (t)^{-1/2}\tau(q))^2}{4\sigma(q)} - \frac{\omega(q)}{t} \tag{10}$$

The corresonding Euler equation is actially P^V however it is not completely equal to traditional form of P^V. In order to find and hence resolve the discrepancy we perform the inverse transformation to variable $q \to qt$. That means returning to traditional form of CHE. We obtain

$$\ddot{q} - \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1}\right)\dot{q}^2 + \frac{\dot{q}}{t} - \frac{1}{2t^2}\left[(c^2+1)\frac{q}{q-1} - d^2\frac{q-1}{q}\right] -$$

$$\frac{1}{2}q(q-1)(2q-1) + \frac{q(q-1)}{t}((c+d)-2a) = 0 \tag{11}$$

The derived equation is a subset of Painlevé equation P^V. The discussion of its generality can be found in [4].

References

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