Factorization Method for the Second-Order Linear Nonlocal Difference Equations

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Abstract. First, we present solvability criteria and a formula for constructing closed-form solutions to arbitrary second-order linear difference equations with variable coefficients and nonlocal multipoint boundary conditions. Next, we develop an operator factorization method for solving exactly boundary value problems for second-order linear difference equations with polynomial coefficients and containing up to the three boundary points. Of particular relevance here are the references [1, 2, 3].

1. Introduction

Denote by S the linear space of all real-valued functions (sequences) $u_k = u(k)$, $k \in \mathbb{N}$. Let $A : S \to S$ be a second-order linear difference operator defined by

$$Au_k = u_{k+2} + a_k u_{k+1} + b_k u_k, \tag{1.1}$$

where $a_k, b_k, u_k \in S$ and $b_k \neq 0$ for all $k \geq k_1$ or preferably for $k = 1, \ldots$. In addition, let the operator $\widehat{A} : S \to S$ be defined as

$$\widehat{A}u_k = Au_k,$$
$$D(\widehat{A}) = \{u_k \in S : \mu_{i1}u_1 + \mu_{i2}u_2 + \ldots + \mu_{i,l}u_l = \beta_i,\ i = 1, 2,\ l \geq 2\}, \tag{1.2}$$

where μ_{ij}, $\beta_i \in \mathbb{R}$, $i = 1, 2$, $j = 1, \ldots, l$; that is to say $\widehat{A}$ is a restriction of A denoted compactly by $\widehat{A} \subset A$.

Let $u_k^{(1)}, u_k^{(2)}$ be a fundamental solution set of the homogeneous equation $Au_k = 0$ and $u_k^{(f_k)}$ be a particular solution of the non-homogeneous equation

$Au_k = f_k,\ f_k \in S$. Introduce the vector $\mathbf{u}_k^{(H)} = (u_k^{(1)}\ u_k^{(2)})$ and the associated Casorati matrix along with the vectors

$$C_0 = \begin{pmatrix} u_1^{(1)} & u_1^{(2)} \\ u_2^{(1)} & u_2^{(2)} \end{pmatrix}, \quad \mathbf{u}_0 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathbf{u}_0^{(f_k)} = \begin{pmatrix} u_1^{(f_k)} \\ u_2^{(f_k)} \end{pmatrix}. \tag{1.3}$$

Furthermore, consider the equation $\widehat{A}u_k = f_k$ for $k = 1, \dots l-3$ together with the two nonlocal boundary conditions and define the $l \times l$ matrix

$$D = \begin{pmatrix} b_1 & a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b_2 & a_2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \cdots & \cdots & b_{l-2} & a_{l-2} & 1 \\ \mu_{11} & \mu_{12} & \cdots & \cdots & \cdots & \mu_{1,l-2} & \mu_{1,l-1} & \mu_{1,l} \\ \mu_{21} & \mu_{22} & \cdots & \cdots & \cdots & \mu_{2,l-2} & \mu_{2,l-1} & \mu_{2,l} \end{pmatrix}, \tag{1.4}$$

and the vectors

$$\mathbf{u}_l = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{l-2} \\ u_{l-1} \\ u_l \end{pmatrix} = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} u_3 \\ \vdots \\ u_l \end{pmatrix}, \quad \beta_f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{l-2} \\ \beta_1 \\ \beta_2 \end{pmatrix}. \tag{1.5}$$

Then the following theorem holds.

Theorem 1.1. *If* $\det D \neq 0$, *then* $\mathbf{u}_l = D^{-1}\mathbf{b}_f$ *and the nonlocal boundary value problem*

$$\widehat{A}u_k = f_k \tag{1.6}$$

admits a unique solution which can be obtained in closed-form as

$$u_k = u_k^{(f_k)} + \mathbf{u}_k^{(H)} C_0^{-1}(\mathbf{u}_0 - \mathbf{u}_0^{(f_k)}). \tag{1.7}$$

The application of Theorem 1.1 requires the analytic form of two linearly independent solutions and a particular solution of the corresponding homogeneous and non-homogeneous equations, respectively, which may be very difficult to obtain in many cases with variable coefficients. Alternatively, we can use a factorization method.

2. Factorization Method

Definition 2.1. A second-order linear difference operator A defined by (1.1) is said to be factorable when it can be written as a product (composition) of two first-order linear operators $A_1,\ A_2 : S \to S$, viz.

$$Au_k = A_1 A_2 u_k. \tag{2.1}$$

Lemma 2.2. *An operator A defined by (1.1) is factorable when there exist $r_k,\ s_k \in S$ such that*

$$Au_k = y_{k+1} + r_k y_k, \tag{2.2}$$

$$A_1 y_k = y_{k+1} + r_k y_k, \quad A_2 u_k = y_k, \tag{2.3}$$

where $y_k = u_{k+1} + s_k u_k$. Moreover, $r_k,\ s_k$ are a solution of the difference equations

$$\begin{aligned} s_{k+1} + r_k &= a_k, \\ s_k r_k &= b_k. \end{aligned} \tag{2.4}$$

We confine our investigations to the cases where the coefficients $a_k,\ b_k$ are polynomials and there exist polynomials $r_k,\ s_k$ which satisfy the system of equations (2.4).

Theorem 2.3. *Let $a_k,\ b_k$ be polynomials of degree $Deg\, a_k$ and $Deg\, b_k$, respectively. Then the second-order operator A is factorable in the following cases:*

(i) *If $Deg\, a_k < Deg\, b_k$ and there exists a polynomial s_k of degree $Deg\, s_k = 0\ or\ 1 \ldots or\ Deg\, b_k$ satisfying the equation*

$$s_k s_{k+1} - a_k s_k + b_k = 0, \tag{2.5}$$

or

(ii) *If $Deg\, a_k = Deg\, b_k$ and there exists a polynomial s_k of degree $Deg\, s_k = 0 \quad or \quad Deg\, s_k = Deg\, b_k \quad$ satisfying Eq. (2.5),*

Then the polynomial s_k can be constructed by the method of undetermined coefficients and thus $r_k = a_k - s_{k+1}$.

Now we state the main theorem in this paper.

Theorem 2.4. *Let the second-order linear difference operator $\widehat{A}$ defined by (1.2) with $l = 3$, viz.*

$$\begin{aligned} &\widehat{A}u_k = u_{k+2} + a_k u_{k+1} + b_k u_k, \\ &D(\widehat{A}) = \{u_k \in S :\ \mu_{i1}u_1 + \mu_{i2}u_2 + \mu_{i3}u_3 = \beta_i,\ i = 1, 2\}. \end{aligned} \tag{2.6}$$

Further, let r_k, s_k solve the system of difference equations (2.4). If

$$\det D = \begin{pmatrix} b_1 & a_1 & 1 \\ \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \end{pmatrix} \neq 0, \tag{2.7}$$

then,

(i) *The operator $\widehat{A}$ can be factored to $\widehat{A} = \widehat{A}_1\widehat{A}_2$ where the injective first-order operators $\widehat{A}_1$ and $\widehat{A}_2$ are defined by*

$$\widehat{A}_1 y_k = \quad y_{k+1} + r_k y_k = f_k, \quad D(\widehat{A}_1) = \{y_k \in S : y_1 = u_2^* + s_1 u_1^*\}, \tag{2.8}$$

$$\widehat{A}_2 u_k = \quad u_{k+1} + s_k u_k = y_k^*, \quad D(\widehat{A}_2) = \{u_k \in S : u_1 = u_1^*\}, \tag{2.9}$$

where $y_k = u_{k+1} + s_k u_k$, $\widehat{A}u_k = \widehat{A}_1 y_k$, $\mathbf{u}_3^* = col(u_1^*, u_2^*, u_3^*)$, $\mathbf{b}_f = col(f_1, \beta_1, \beta_2)$ and $\mathbf{u}_3^* = D^{-1}\mathbf{b}_f$, and $y_k^* = \widehat{A}_1^{-1} f_k$.

(ii) *The unique solution of the three-point boundary value problem is given in closed-form by*

$$u_k = \widehat{A}^{-1} f_k = \widehat{A}_2^{-1}\widehat{A}_1^{-1} f_k = \widehat{A}_2^{-1} y_k^*. \tag{2.10}$$

Finally, we solve the next example problem.

Example 2.5. The operator $\widehat{A} : S \to S$ defined by

$$\begin{aligned} &\widehat{A}u_k = u_{k+2} - (k+2)u_{k+1} + (k+1)u_k = (k+1)!\,, \\ &D(\widehat{A}) = \{u_k \in S :\; u_1 - u_2 + 2u_3 = 4, \;\; 2u_1 + u_2 + u_3 = 5\}, \end{aligned} \tag{2.11}$$

is injective and the unique solution of (2.11) is given by the formula

$$u_k = \frac{5}{4} + \sum_{j=1}^{k-1} j!\Big(j - \frac{3}{2}\Big) \tag{2.12}$$

3. Conclusion

The technique presented here is simple to use, it can be easily incorporated to any Computer Algebra System (CAS) and more important it can be extended to deal with more complicated problems embracing nonlocal boundary conditions with many points and non-polynomial variable coefficients.

References

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