Local and Global Integrability of ODEs

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Abstract. We consider autonomous planar systems of ordinary differential equations with a polynomial nonlinearity. These systems are resolved with respect to derivatives and can contain free parameters. To study local integrability of the system near each stationary points, we use an approach based on Power Geometry[1] and on the computation of the resonant normal form[2, 3]. For the pair of concrete planar systems[4] and[5], we found the complete set of necessary conditions on parameters of the system for which the system is locally integrable near each stationary points. The main idea of this report is in the hypothesis that if for each fixed set of parameters such that all stationary points of the equation are centers then this system has the global first integral of motion. So from some finite set of local properties we can obtain a global property. But if the system has some invariant lines or separatists, this first integral can exist only in the part of the phase space, where center points take place.

Full Phase Space Integrability

Firstly we studied the system which is a partial case of the system[4]

$$\begin{aligned} \frac{dx}{dt} &= -y^3 - b\,x^3 y + a_0\,x^5 + a_1\,x^2 y^2, \\ \frac{dy}{dt} &= \frac{1}{b}\,x^2 y^2 + x^5 + b_0\,x^4 y + b_1\,x\,y^3. \end{aligned} \tag{1}$$

Thus, we consider the system with five arbitrary parameters a_i, b_i, $(i = 0, 1)$ and $b \neq 0$. After the power transformation

$$x = u\,v^2, \qquad y = u\,v^3,$$

we obtained the system in the form

$$\begin{aligned} \frac{du}{d\tau} &= -3\,u - [3\,b + (2/b)]u^2 - 2\,u^3 + (3\,a_1 - 2\,b_1)u^2 v + (3\,a_0 - 2\,b_0)u^3 v\,, \\ \frac{dv}{d\tau} &= v + \left[b + \frac{1}{b}\right] u\,v + u^2 v + (b_1 - a_1)u\,v^2 + (b_0 - a_0)u^2 v^2\,. \end{aligned} \tag{2}$$

The point $x = y = 0$ blows up into two straight invariant lines $u = 0$ and $v = 0$. Along the line $u = 0$, the system has the stationary point $u = v = 0$. Along the second line $v = 0$. So if $b^2 \neq 2/3$, this system has four elementary stationary points[6]

$$u = 0, \qquad u = -\frac{1}{b}, \qquad u = -\frac{3b}{2}, \qquad u = \infty\,.$$

At the each point above there exist values of parameters when the system is locally integrable, but if $b^2 \neq 2/3$ there are only 4 two dimensional combinations, where local integrability takes place simultaneously

$$\begin{array}{lll} 1) & a_0 = 0, \quad a_1 = -b_0\,b, \quad b_1 = 0, & \\ 2) & b_1 = -2\,a_1, & a_0 = a_1 b, \quad b_0 = b_1 b, \\ 3) & b_1 = (3/2)\,a_1, & a_0 = a_1 b, \quad b_0 = b_1 b, \\ 4) & b_1 = (8/3)\,a_1, & a_0 = a_1 b, \quad b_0 = b_1 b. \end{array} \tag{3}$$

In [7], we have calculated first integrals of the system (2) for all cases (3) (mainly by the Darboux method, see, e.g., [8]). These integrals are

$$I_1(x,y) = 2\,x^3 + 3\,b\,y^2,$$
$$I_2(x,y) = 2\,x^3 - 6\,a_1\,b\,x^2\,y + 3\,b\,y^2,$$

$$I_3(x,y) = \frac{a_1 x^2 \left(-4 + 3^{5/6}\,{}_2F_1\left(2/3, 1/6; 5/3; -2\,x^3/(3\,b\,y^2)\right) \times \left(3 + 2x^3/(b\,y^2)\right)^{1/6}\right)}{y^{4/3}(3\,b + 2\,x^3/y^2)^{1/6}} + \frac{4y}{y^{4/3}(3\,b + 2\,x^3/y^2)^{1/6}},$$

$$I_4(u,v) = \frac{u\,(3 + 2\,a_1^2 b u) + 6\,a_1\,b\,v}{3\,u\left[u^3(6 + a_1^2 b\,u) + 6\,a_1^2 b\,u^2 v + 9\,b\,v^2\right]^{1/6}} - 8\,a_1\sqrt{-b}/3^{5/3} B_{6 + a_1\sqrt{-6\,b\,u} + 3\,v\sqrt{-6\,b/u^3}}(5/6, 5/6)\,,$$

where $B_t(a,b)$ is the incomplete beta function and ${}_2F_1(a,b;c;z)$ is the hypergeometric function [9].

In the case $b^2 = 2/3$ the situation is a bit more complicate and we have at least two additional stationary points which are compatible with (3) at $b^2 = 2/3$
5. $b_1 = 3a_1/2, \quad a_0 = (2b_0 + b(3a_1 - 2b_1))/3,$
6. $b_1 = 6a_1 + 2\sqrt{6}b_0, \quad a_0 = (2b_0 + b(3a_1 - 2b_1))/3,$ and two more global first

integrals of a motion for these values of parameters exist

$$I_5(x,y) = \frac{y}{x^2}\left(\sqrt{6}+\frac{2x^3}{y^2}\right)^{-7/6}\left(\frac{x^3}{y^2}\right)^{2/3}\times\Big\{42\sqrt{6}+\frac{1}{xy^3}\Big[-36a_1x^6-16\sqrt{6}b_0x^6 + 84x^4y-24\sqrt{6}a_1x^3y^2-36b_0x^3y^2+2^{1/3}\left(\frac{x^3}{y^2}\right)^{1/3}y^2\cdot\left(\sqrt{6}+\left(\frac{x^3}{y^2}\right)^{2/3}\right)\times (2\big(9a_1+4\sqrt{6}b_0\big)x^3+3\big(3\sqrt{6}a_1+8b_0\big)y^2\Big)\times {}_2F_1\left(-1/2,1/3;1/2;\tfrac{3y^2}{3y^2+\sqrt{6}x^3}\right)\Big]\Big\},$$

$$I_6(x,y) = y\cdot\left(\sqrt{2/3}+\frac{x^3}{y^2}\right)^{-\frac{1}{2}+\frac{a_1}{-6a_1-2\sqrt{6}b_0}}\left(\frac{x^2}{y}\right)^{-\frac{a_1}{3a_1+\sqrt{6}b_0}}\times \left\{3+\frac{x^2}{y^2}\left[\sqrt{6}x+3\left(2a_1+\sqrt{6}b_0\right)y\right]\right\}.$$

Partial Phase Space integrability

We have examples when integrability takes place only in a part of the phase space. Let us see as the system

$$\begin{aligned}\frac{dx}{dt} &= \quad y+2xy,\\ \frac{dy}{dt} &= - \quad x-bx^2+cxy+y^2.\end{aligned} \tag{4}$$

This system has 3 different stationary points

$$\begin{aligned}&x=0, y=0,\\ &x=-(1/b), y=0,\\ &x=-1/2, y=(c-\sqrt{-4b-c^2-7})/4.\end{aligned}$$

The local integrability (i.e the center case) take place in the first case at $b=1$ or $c=0$ only. At this values the second and third cases are local nonintegrable cases (focuses), at that the third case lies in complex values of y. The corresponding first integrals are

$$\begin{aligned}&\text{At } b=1; \quad I(x,y) = (1+2x)^{(}\frac{1}{4}(-4+c(c+\sqrt{c^2-4})),\\ &\text{At } c=0; \quad I(x,y) = (-1-2bx(1+x)-2y^2+(b-1)(2x+1)\log 2x+1)/(8x+4).\end{aligned} \tag{5}$$

In both cases we have the invariant line $x=-1/2$ which separates the left part, where exists the global first integral (center case) from right part side where integrability does not exist.

Conclusion

We propose the hypothesis that the local integrability in all stationary points of autonomous planar systems of ordinary differential equations with polynomial right sides and resolved with respect to derivatives leads to existence of the global first integral if this system has these local integrability at the same sets of parameters. If the system has invariant curves or separatrices such integral can exist only in a part of the phase space.

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