On Strongly Consistent Finite Difference Approximations

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Solving partial differential equations (PDEs) belongs to the most fundamental and practically important research challenges in mathematics and in the computational sciences. Such equations are typically solved numerically since obtaining their explicit solution is usually very difficult in practice or even impossible. One of the classical and nowadays well-established and popular approaches is the finite difference method [1, 2, 3] which exploits a local Taylor expansion to replace a differential equation by the difference one. This raises the question how to preserve fundamental properties of the underlying PDEs at the discrete level. From a geometric point of view, the most important properties are symmetries and conservation laws. Importance of conservation laws in mathematical physics could not be underestimated, since many fundamental properties for nonlinear PDEs (like existence and uniqueness of solutions) typically are based on conservation laws. From algebraic perspective, the basic object which should be preserved is algebraic relations between equations and their differential (difference) consequences. The problem here occurs because finite difference approximation of derivation doesn't satisfy Leibnitz rule.

The fundamental requirement of a finite difference scheme (FDS) is its convergence to a solution of the corresponding differential problem as the grid spacings go to zero. According to the Lax-Richtmyer equivalence theorem [4, 5], for a scalar PDE it has been adopted that the convergence is provided if a given finite-difference approximation (FDA) to the PDE is consistent and stable. The consistency implies a reduction of the FDA to the original PDE when the grid spacings go to zero, and it is obvious that the consistency is necessary for convergence. The theorem states that a FDS for an initial value (Cauchy) problem providing the existence and uniqueness of the solution converges if and only if its FDA is consistent and numerically stable.

In this talk we describe algorithmic methods to generate FDAs to PDEs on orthogonal and uniform grids, and to verify strong consistency of the obtained FDAs. The main algorithmic tool for the case of linear PDEs is the difference elimination provided by Groebner bases [6, 7, 8] for a certain elimination ranking.

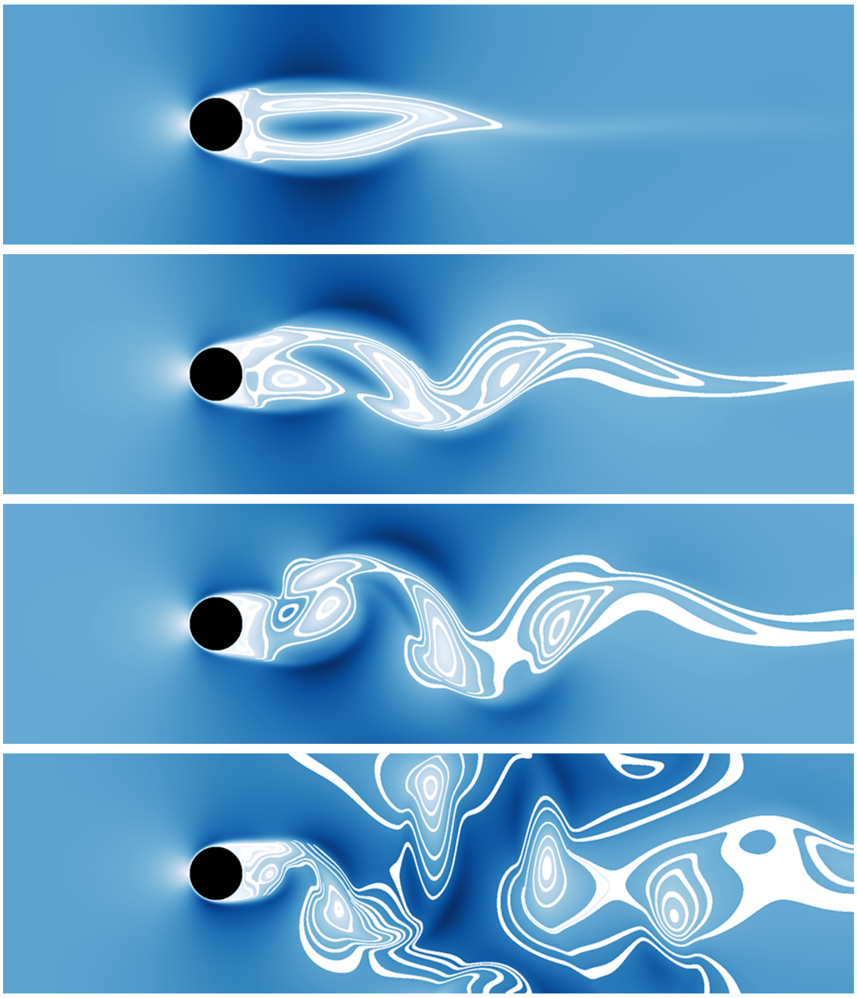

FIGURE 1. Simulation of the Kármán vortex street computed with the new FDA. The characteristic repeating pattern of swirling vortices can be observed, cf. [15].

Given a system of polynomially-nonlinear PDEs and its FDA, the s-consistency analysis is based on a computation of a difference standard Groebner basis and the construction of a differential Thomas decomposition [9, 10] for the PDE system. This talk is an extension of the methodology of [8, 11, 12, 13, 14]. As a relevant example in practice, we apply the procedure of the strong consistent FDA generation to the two-dimensional Navier-Stokes equations for the unsteady motion of an incompressible fluid of constant viscosity. For these equations, we construct two fully conservative FDAs (one s-consistent and one w-consistent). We use the FDAs for the numerical simulation on exact solutions and consider a Kármán vortex street to analyze the influence of the consistency on the numerical quality of these schemes.

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