Bifurcation diagrams for polynomial nonlinear ordinary differential equations

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Field of study

- $y''_{xx} + \lambda f\big(y(x)\big) = 0, \; where \; x \in (0,1)$
- $y'_x(0) = 0, \; y(1) = 0$

- $\lambda > 0$ – bifurcation parameter
- $f(y)$ – nonlinear function
  - $f(y) = (y - a_1)(y - a_2)(y - a_3) \ldots (y - a_{2n-1})(a_{2n} - y)$

- bifurcation curve and diagram
- P. Korman - Y. Li - T. Ouyang Theorem

?

- Exact number of positive solutions
- How the solutions change due changing the bifurcation parameter λ

BVP

- $y''_{xx} + \lambda f\big(x, y(x)\big) = 0$, $where\ x \in (a, b)$, λ>0
- $y(a) = y(b) = 0$

Solutions:

- $y = y(x, \lambda)$

?

- Exact number of positive solutions
- How the solutions change due changing the bifurcation parameter λ

Autonomous 2nd order ODE

- $y''_{xx} + \lambda f\big(y(x)\big) = 0,\ where\ x \in (a, b)$
- $y(a) = y(b) = 0$

Positive solutions

- $y''_{xx} + \lambda f\big(y(x)\big) = 0,\ where\ x \in (-1,1)$
- $y(-1) = y(1) = 0$

Even number of zeros of solutions

- $y''_{xx} + \lambda f\big(y(x)\big) = 0,\ where\ x \in (0,1)$
- $y'_{x}(0) = 0,\ \ y(1) = 0$

Korman P., Global solution branches and exact multiplicity of solutions for two point boundary value problems.

Auxiliary problem

- $y(0) = a$ – maximal value of the solution of the BVP, $a > 0$
- $y''_{tt} + f(y) = 0,\ where\ x \in (0,1)$
- $y'_t(0) = 0, y(0) = a$
- $y'_t = \sqrt{2[F(a) - F(y)]},\ where\ F(y) = \int_0^y f(t)dt.$
- $\int_0^a \frac{dt}{\sqrt{F(a)-F(y)}} = \sqrt{2}(1 - t).$
- $\lambda(a) = \frac{1}{2}\left(\int_0^a \frac{dt}{\sqrt{F(a)-F(y)}}\right)^2$ - bifurcation curve

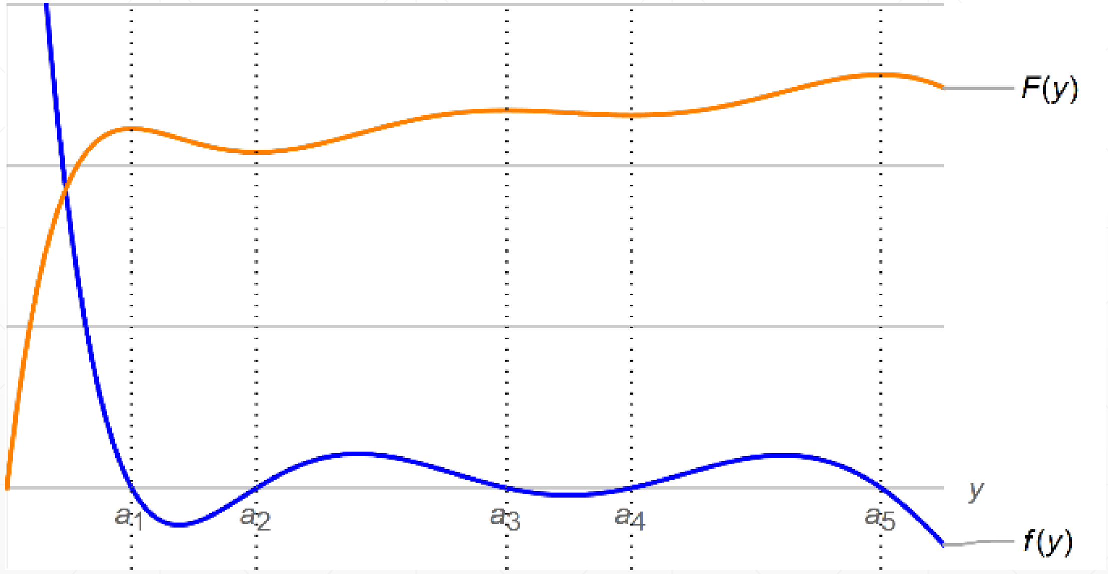
Bifurcation curve $\lambda(a)$

- $\lambda = \lambda(a)$ - bifurcation **curve**;
- Turning points of $\lambda(a)$ – bifurcation **points**;
  - $\lambda'(a) = 0$
- Plot of function $\lambda = \lambda(a)$ – bifurcation **diagram**, implying an image of the change in the possible dynamic modes of the system with a change in the value of bifurcation parameter λ.

$$\lambda(a) = \frac{1}{2}\left(\int_0^a \frac{dt}{\sqrt{F(a)-F(y)}}\right)^2$$ - bifurcation curve

$f(y)$ as a polynomial of odd degree

- $f(y) = (y - a_1)(y - a_2)(y - a_3) \dots (y - a_{2n-2})(a_{2n-1} - y)$
  - $0 \le a_1 < a_2 < \dots < a_{2n-2} < a_{2n-1}$
  - $F(a_1) < F(a_2) \dots < F(a_{2n-2}) < F(a_{2n-1})$



$$y''_{xx} + \lambda f(y) = 0, \; where \;\; x \in (0,1)$$
$$y'_x(0) = 0, \;\; y(1) = 0$$

!

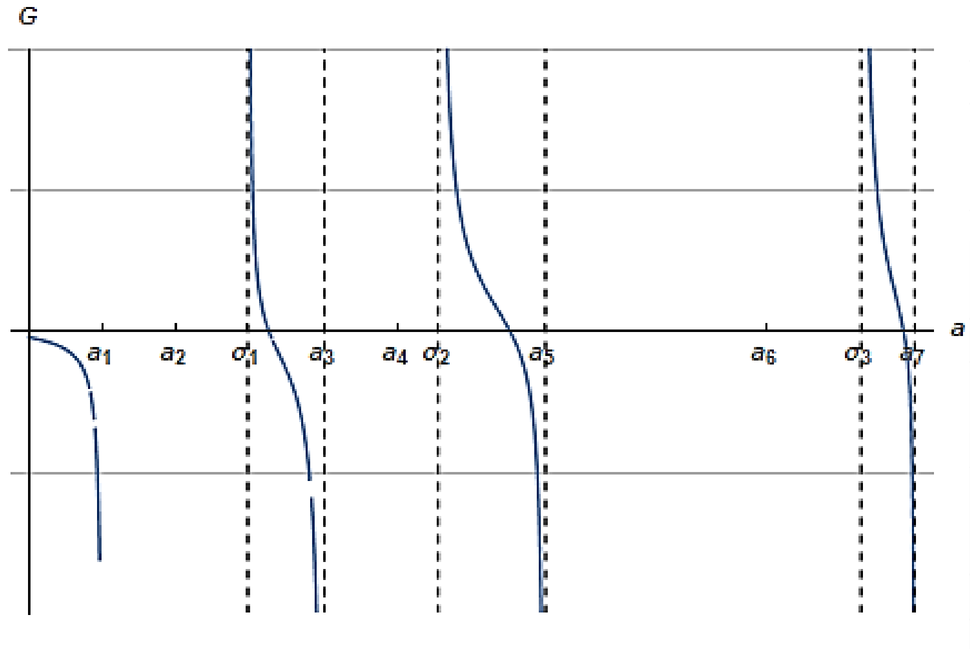
- Trivial solutions of BVP:

  $$\boldsymbol{y = a_i}$$
- $f(y) > 0$ on $(a_{2n-2}; a_{2n-1})$
- $f(y) < 0$ on $(a_{2n-3}; a_{2n-2})$

P. Korman, Y. Li and T. Ouyang Theorem

- A solution of the BVP with the maximal value $a = y(0)$ is singular if and only if

$$G(a) \equiv \sqrt{F(a)} \int_0^a \frac{f(a)-f(\tau)}{[F(a)-F(\tau)]^{3/2}} - 2 = 0, \text{ where } F(y) = \int_0^y f(y)dy$$



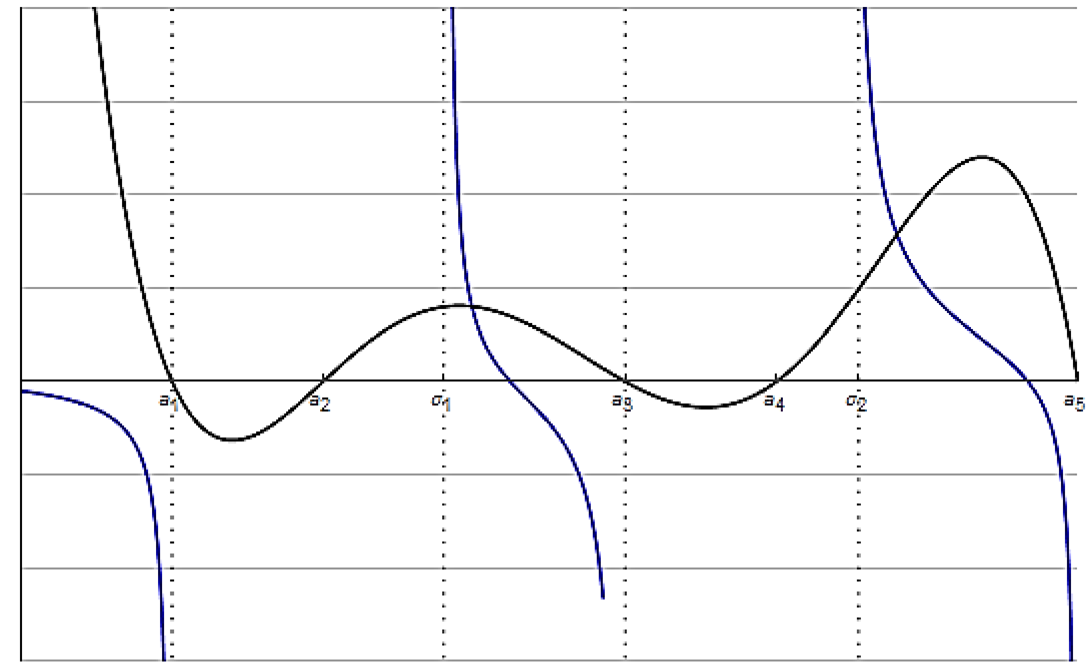
Korman P., Li Y., Ouyang T. Computing the location and the direction of bifurcation. Math. Research Letters // 2005.

$$\mathbf{G(a)} \equiv \sqrt{\boldsymbol{F(a)}} \int_0^a \frac{\boldsymbol{f(a)-f(t)}}{[\boldsymbol{F(a)-F(t)}]^{3/2}} - \mathbf{2} = \mathbf{0}, F(y) = \int_0^y f(y)dy$$

- $\lim_{a \to a_1^-} G(a) = -\infty$,
  - to the left of a_1

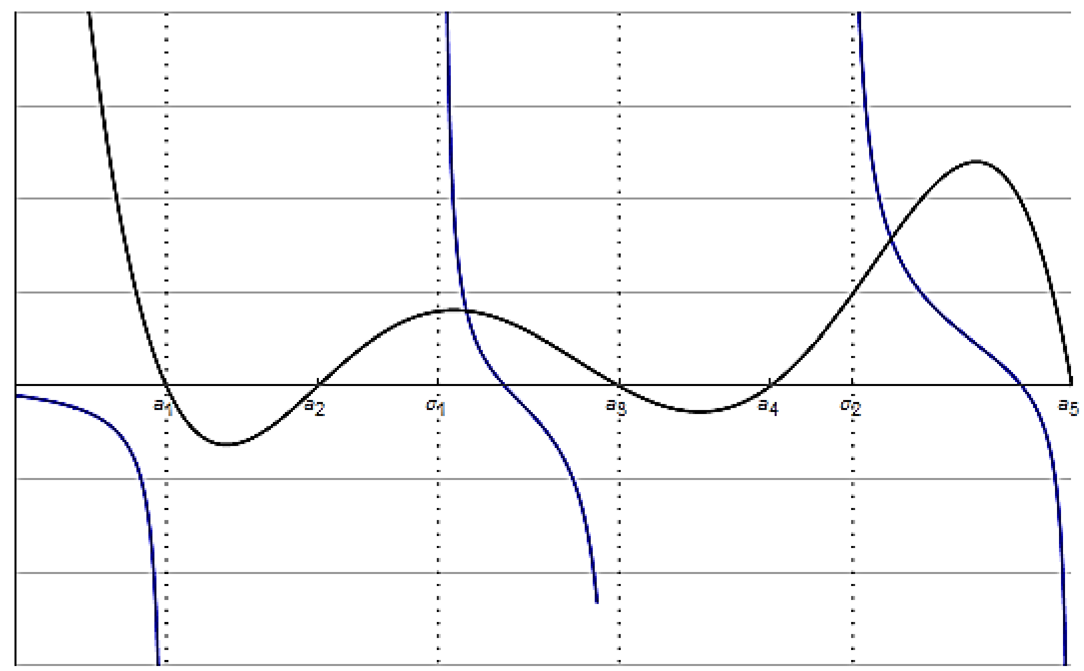
  there is no bifurcation point.
- $\lim_{a \to a_{2n-1}^-} G(a) = -\infty$,
  - to the left of a_{2n-1}

  there is no bifurcation point.
- $\lim_{a \to \sigma_{n-1}^+} G(a) = +\infty$,
  - where $\sigma_{n-1} \in (a_{2n-2}, a_{2n-1})$
  - $\int_{a_{2n-3}}^{\sigma_{n-1}} f(s)ds = 0$
  - to the right of σ_{n-1} there is no bifurcation point.



$$\mathbf{G(a)} \equiv \sqrt{F(a)} \int_0^a \frac{f(a)-f(t)}{[F(a)-F(t)]^{3/2}} - 2 = 0, \quad F(y) = \int_0^y f(y)dy$$

- $G(a) = 0$ only on the intervals:
  - (a_2, a_3), (a_4, a_5), (a_6, a_7) … (a_{2n-2}, a_{2n-1});
  - Only these intervals contain bifurcation points.

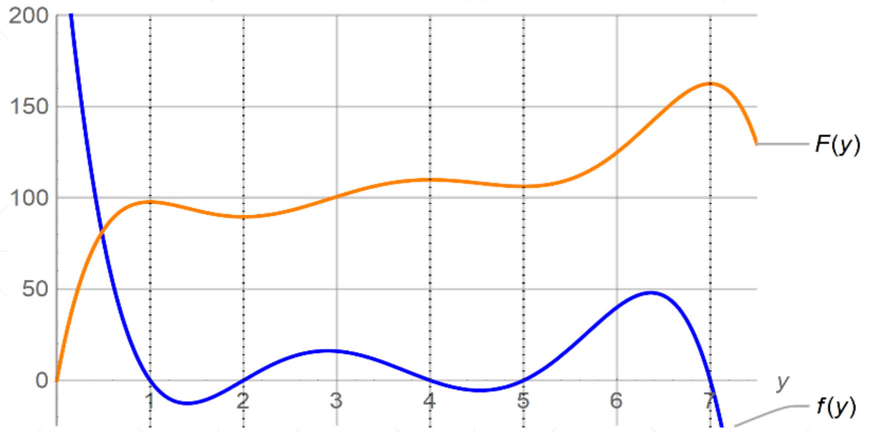


Wolfram Mathematica 11.0

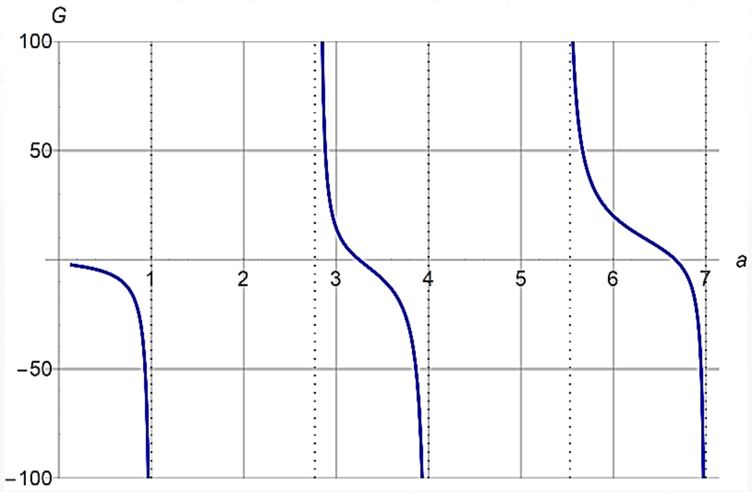
- NDSolve
- ParametricNDSolve
- Nintegrate
- $\lambda(a)$ and $G(a)$ using NumericQ
- NMinimize / NMaximize / FindRoot for $\lambda'(a) = 0$

Ex1. Polynomial of 5th degree.
$f(y) = (y-1)(y-2)(y-4)(y-5)(7-y)$

- $y''_{xx} + \lambda f(y) = 0$, where $x \in (0,1)$
- $y'_x(0) = 0,\ y(1) = 0$



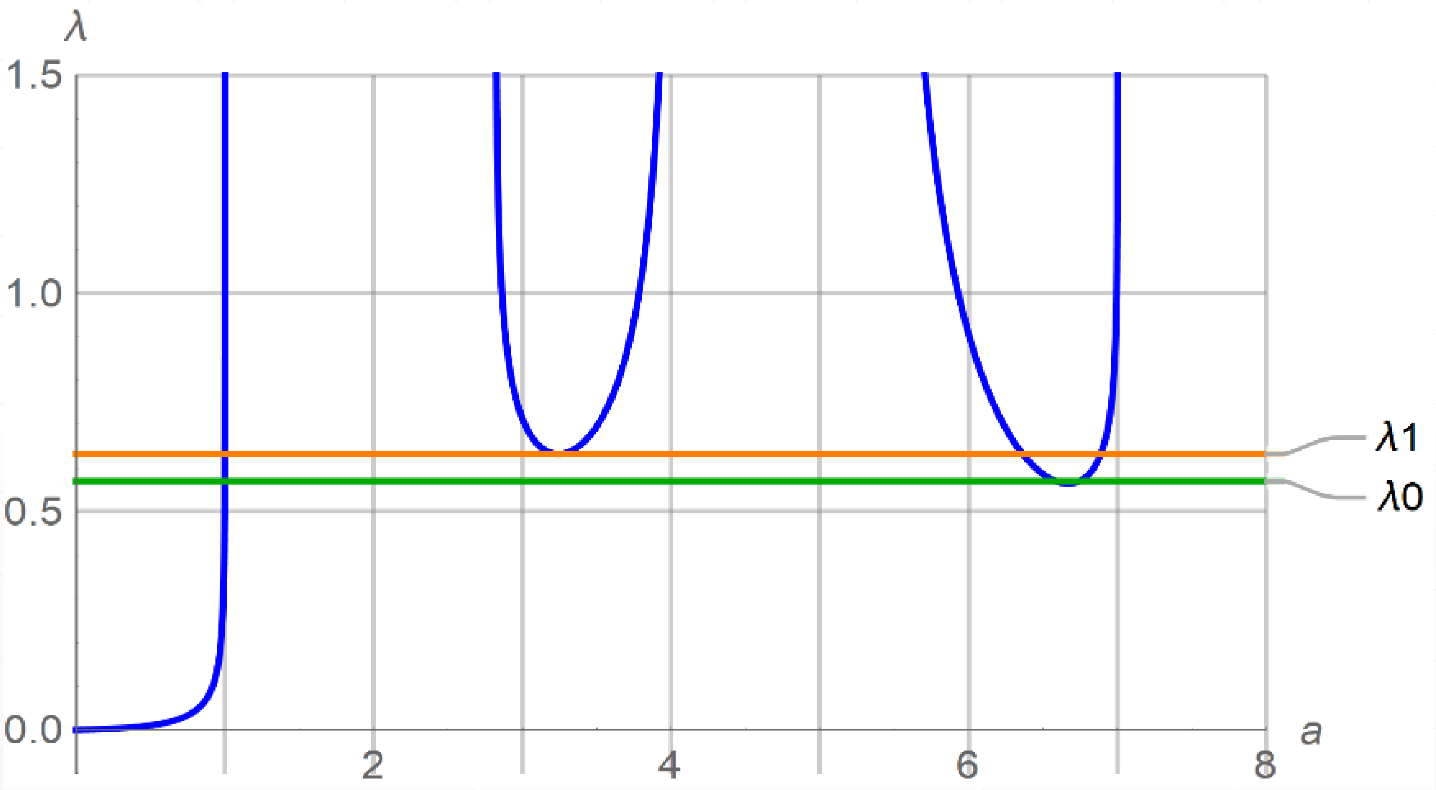
- $f(y) > 0$ on $(2;4)$ and $(5;\ 7)$
- $F(1) < F(2) < F(4) < F(5) < F(7)$
- $G(a) = 0$ on $(3;\ 3.5)$
- $G(a) = 0$ on $(6.5;\ 7)$



Ex1. Polynomial of 5th degree. $f(y) = (y-1)(y-2)(y-4)(y-5)(7-y)$

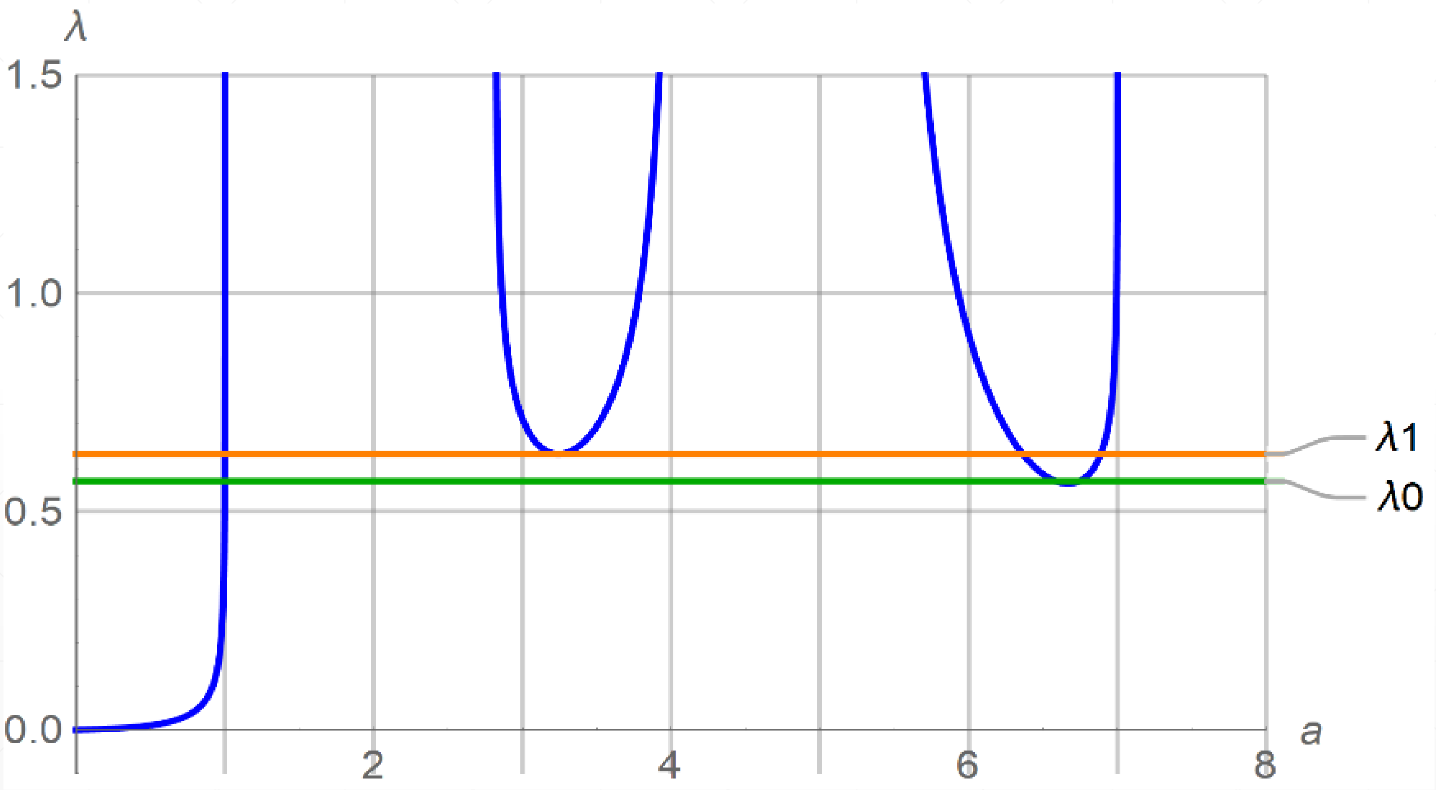
- $y''_{xx} + \lambda f(y) = 0, \ where\ x \in (0,1)$
- $y'_x(0) = 0, \ y(1) = 0$

- $G(a) = 0$ on $(3;\ 3.5)$
- $G(a) = 0$ on $(6.5;\ 7)$
- $\lambda(a) = \frac{1}{2}\left(\int_0^a \frac{dt}{\sqrt{F(a)-F(y)}}\right)^2$
  - $\lambda'(a) = 0 \rightarrow$
  - $a_0 = 3.2417$ where $\lambda_0 = 0.63210$
  - $a_1 = 6.5866$ where $\lambda_1 = 0.56973$



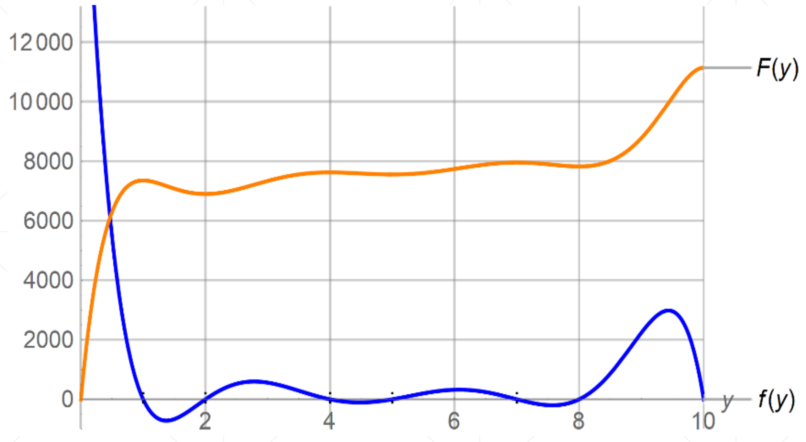
Ex1. Polynomial of 5th degree. $f(y) = (y-1)(y-2)(y-4)(y-5)(7-y)$

- Trivial solutions $y = a_i$:
  - $y = 1$
  - $y = 2$
  - $y = 4$
  - $y = 5$
  - $y = 7$
- $0 < \lambda < \lambda_0$: 1 solution
- $\lambda = \lambda_0$: 2 solutions
- $\lambda_0 < \lambda < \lambda_1$: 3 solutions
- $\lambda = \lambda_1$: 4 solutions
- $\lambda > \lambda_1$: 5 solutions

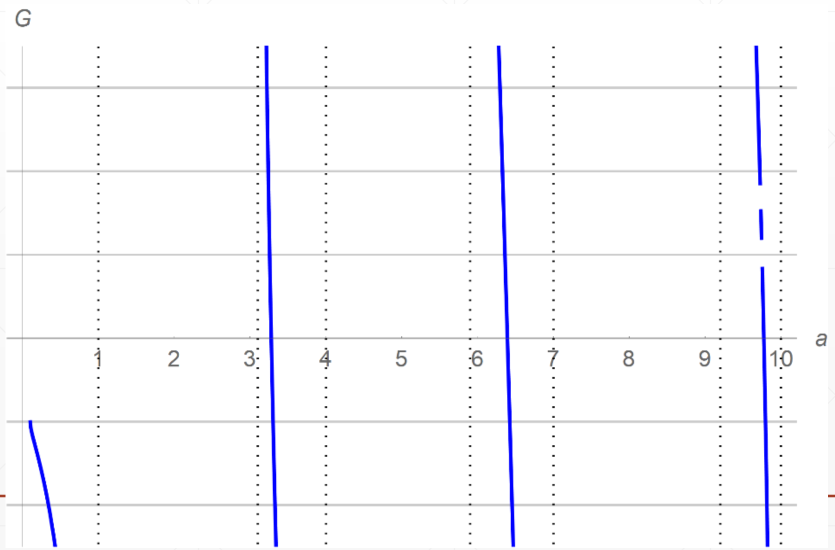


Ex2. Polynomial of 7th degree. $f(y) = (y-1)(y-2)(y-4)(y-5)(y-7)(y-8)(10-y)$

- $y''_{xx} + \lambda f(y) = 0$, where $x \in (0,1)$
- $y'_x(0) = 0,\ y(1) = 0$



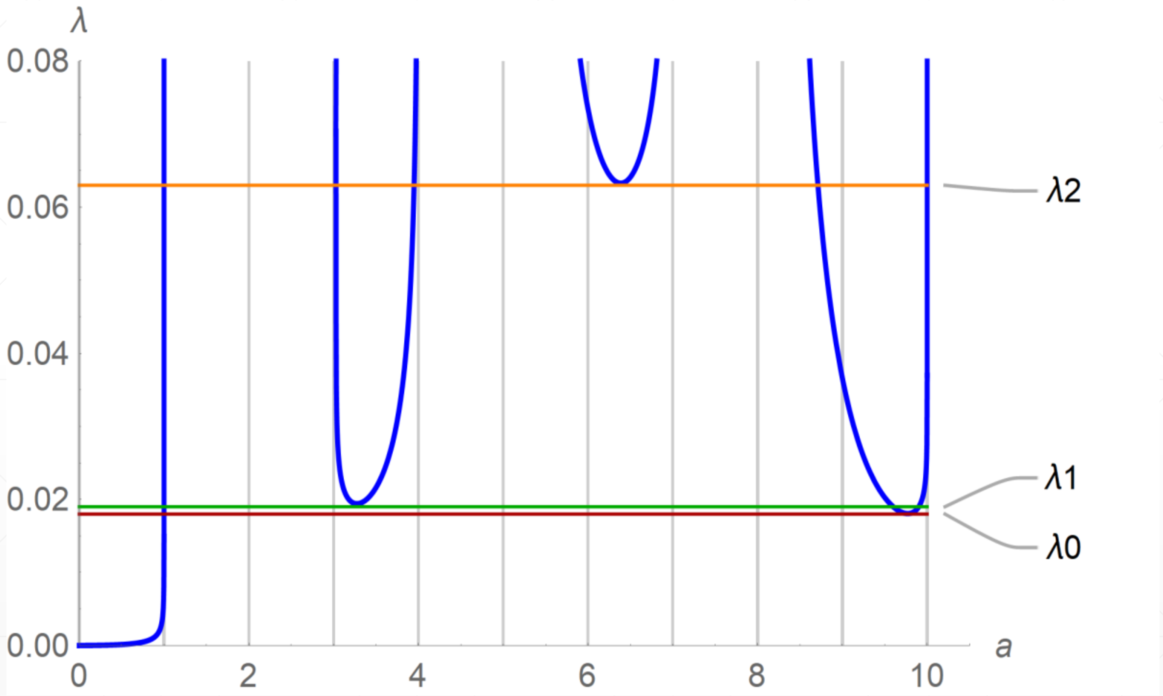
- $f(y) > 0$ on $(2;4)$, $(5;\ 7)$ and $(8;10)$
- $F(1) < F(2) < F(4) < F(5) < F(7) < F(8) < F(10)$
- $G(a) = 0$ on $(3;\ 3.5)$
- $G(a) = 0$ on $(6.5;\ 7)$
- $G(a) = 0$ on $(9.5;10)$



Ex2. Polynomial of 7th degree.
$f(y) = (y-1)(y-2)(y-4)(y-5)(y-7)(y-8)(10-y)$

- $y''_{xx} + \lambda f(y) = 0$, where $x \in (0,1)$
- $y'_x(0) = 0,\ y(1) = 0$

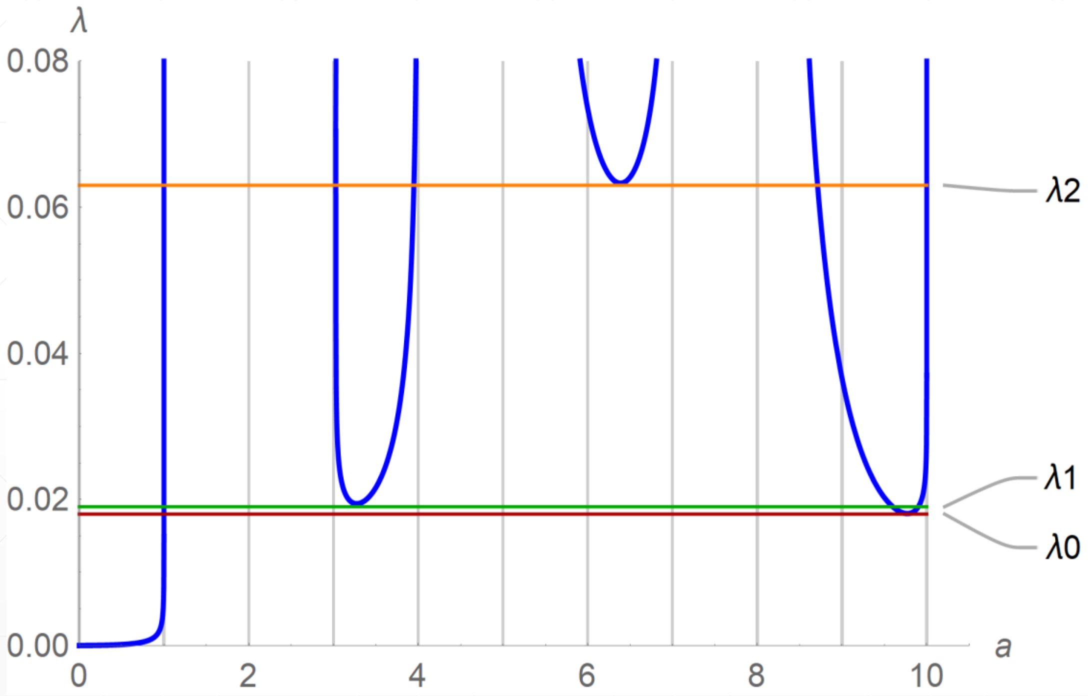
- $G(a) = 0$ on $(3;\ 3.5)$
- $G(a) = 0$ on $(6.5;\ 7)$
- $G(a) = 0$ on $(9.5; 10)$
- $\lambda(a) = \frac{1}{2}\left(\int_0^a \frac{dt}{\sqrt{F(a)-F(y)}}\right)^2$
  - $\lambda'(a) = 0 \rightarrow$
  - $a_0 = 3.2417$ where $\lambda_0 = 0.0194$
  - $a_1 = 6.38791$ where $\lambda_1 = 0.0633$
  - $a_2 = 9.6693$ where $\lambda_2 = 0.0181$



Ex2. Polynomial of 7th degree.

$$f(y) = (y-1)(y-2)(y-4)(y-5)(y-7)(y-8)(10-y)$$

- Trivial solutions $y = a_i$:
  - $y = 1$
  - $y = 2$
  - $y = 4$
  - $y = 5$
  - $y = 7$
  - $y = 8$
  - $y = 10$
- $0 < \lambda < \lambda_0$: 1 solution
- $\lambda = \lambda_0$: 2 solutions
- $\lambda_0 < \lambda < \lambda_1$: 3 solutions
- $\lambda = \lambda_1$: 4 solutions
- $\lambda_1 < \lambda < \lambda_2$: 5 solutions
- $\lambda = \lambda_2$: 6 solutions
- $\lambda > \lambda_2$: 7 solutions



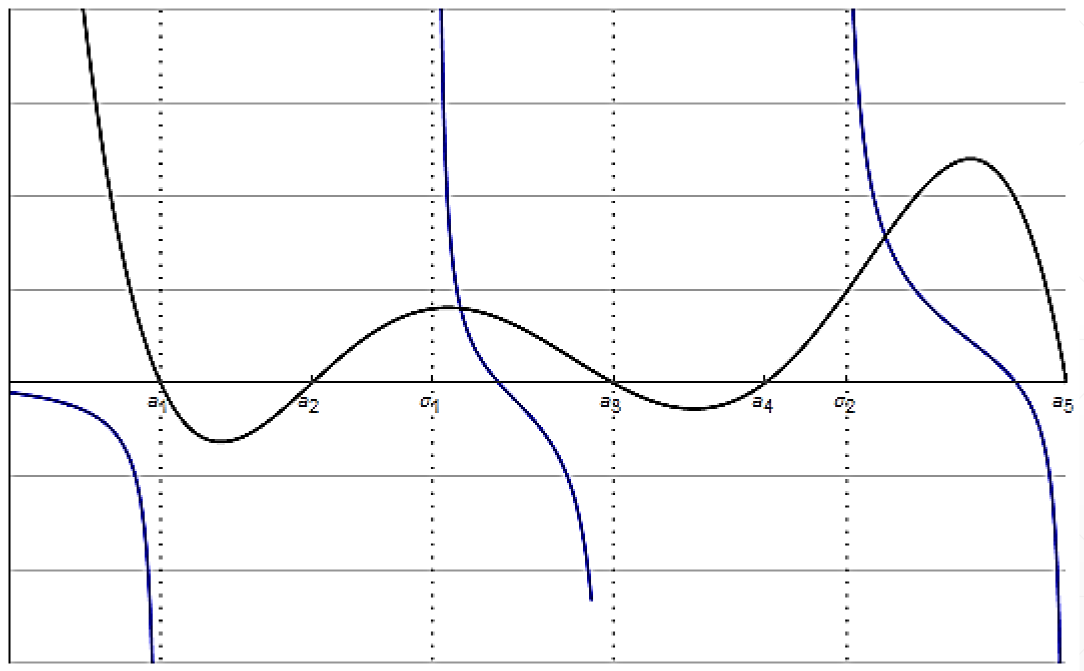
Conclusion

$$f(y) = (y - a_1)(y - a_2)(y - a_3) \dots (y - a_{2n-2})(a_{2n-1} - y)$$

- $y''_{xx} + \lambda f(y) = 0$, where $x \in (0,1)$
- $y'_x(0) = 0,\ y(1) = 0$

- G(a) has zeros on the intervals:
  - (a_2, a_3), (a_4, a_5), (a_6, a_7) … (a_{2n-2}, a_{2n-1})

- Trivial solutions: $y = a_i, i = 1..2n$
- $\lambda = \lambda_n$: 2i solutions
- $\lambda_n < \lambda < \lambda_{n+1}$: 2i-1 solutions



Conclusion

$$f(y) = (y - a_1)(y - a_2) \dots (y - a_{2n-2})(a_{2n-1} - y)$$

- Using properties of G(a):
  - G(a) has zeros only on the intervals: (a_2, a_3), (a_4, a_5), $(a_6, a_7) \dots (a_{2n-2}, a_{2n-1})$ – only these intervals contain bifurcation points;
  - Asymptotic behavior.
- Using Computational methods of numerical integration and differentiation of the system Wolfram Mathematica 11.0.
- **Maximal number of positive solutions depends on the degree of polynomial and equals $\boldsymbol{\lambda_n}$,** so there is one solution to the problem where $\lambda < \lambda_0$, there are two solutions where $\lambda = \lambda_0$, there are three solutions where $\lambda_0 < \lambda < \lambda_1$, etc.

References

- [1] P. Korman, Global Solution Branches and Exact Multiplicity of Solutions for Two Point Boundary Value Problems. Handbook of Differential Equations: Ordinary Differential Equations, 2006.
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- [3] P. Korman, Y. Li, T. Ouyang Computing the location and the direction of bifurcation, Math. Research Letters, 2005.
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- [5] P. Korman, Y. Li, T. Ouyang Verification of bifurcation diagrams for polynomial-like equations, Journal of Computational and Applied Mathematics, 2008.