Bifurcation diagrams for polynomial nonlinear ordinary differential equations

Daria Chemkaeva and Alexandr Flegontov

Abstract. This study considers the general case for classes of nonlinear boundary value problems for a second-order autonomous ordinary differential equation with homogeneous boundary conditions. The general case is studied applying to polynomial-like nonlinearities. We investigate the number of positive solutions to the problem. The research is confirmed by computer-generated function of P. Korman, Y. Li, T. Ouyang Theorem and bifurcation diagrams.

Introduction

We study the existence of positive solutions of the nonlinear two-point boundary value problem:

$$y''_{xx} + \lambda f(y(x)) = 0, \quad x \in (-1;1), \tag{1}$$

$$y(-1) = y(1) = 0. \tag{2}$$

Assume $f = f(y)$ so second order ODE is autonomous, where parameter λ is positive. In this case, the bifurcation arises when the number of solutions of the differential equation changes as the parameter λ changes.

The problem (1)–(2) describes many physical processes, for example, belongs to the problems of combustion of gases and population dynamics. The nonlinearity of $f = f(y)$ in combustion theory denotes intermediate steady states of the temperature distribution y, and the bifurcation parameter λ determines the amount of unburnt substance.

Section 1 is technical and contains useful supplement of P. Korman, Y. Li and T. Ouyang theorem. Section 2 is the main part of the study where the behavior of function from P. Korman, Y. Li and T. Ouyang Theorem is studied considering that nonlinear function is a polynomial of odd degree with a_{2n-1} roots ($n = 2, \ldots, k, k \geq 2$). The examples are provided with description of the respective time-map functions, solutions, bifurcations and visualizations. Finally, we summarize the results and make conclusions.

1. P. Korman, Y. Li and T. Ouyang Theorem

The differential equation (1) with boundary conditions (2) has k zeros of solutions depending on the bifurcation parameter. Let us consider the case when the number of zeros of the solutions is even. In this case, solutions (1)–(2) are symmetric with respect to $x = 0$ [1], hence (1)–(2) can be reduced to the form:

$$y''_{xx} + \lambda f(y(x)) = 0, \quad x \in (0;1), \tag{3}$$

$$y'_x(0) = 0, \quad y(1) = 0. \tag{4}$$

It is known that positive solutions can be determined using term $y(0) = a$. This zero function is a time-map for solutions (1)–(2) [2] and the maximal value of the solution of the boundary value problem, which uniquely determines the pair $(\lambda, y(x))$. We show that by defining a we can uniquely determine the appropriate value $\lambda > 0$ and the solution of the problem $y = y(x)$.

Suppose that $t = \sqrt{\lambda}x$ so for the function $y = y(t)$ we consider the intermediate Cauchy problem:

$$y''_{tt} + f(y) = 0, \tag{5}$$

$$y'_t(0) = 0, \quad y(0) = a. \tag{6}$$

We use the substitution and find the first integral of equation (5), fulfilling the first boundary condition (6):

$$y'_t = \sqrt{2}\sqrt{F(a) - F(y)}, \quad F(y) = \int_0^y f(y)\,dy.$$

For the existence of the solution (5)–(6) it is necessary to satisfy the inequality $F(a) \geq F(y)$. The solution of boundary value problem (5)–(6) in implicit form is:

$$t = \int_0^y \frac{dt}{\sqrt{F(a) - F(t)}}.$$

Returning to boundary value problem (3)–(4), the bifurcation parameter is:

$$\lambda(a) = \frac{1}{2}\left[\int_0^a \frac{dt}{\sqrt{F(a) - F(t)}}\right]^2. \tag{7}$$

The function $\lambda = \lambda(a)$ is called the bifurcation curve; its turning points are bifurcation points. The plot of this function is called the bifurcation diagram [3], implying an image of the change in the possible dynamic modes of the system with a change in the value of bifurcation parameter λ.

The authors P. Korman, Y. Li and T. Ouyang prove that a solution of the problem (1)–(2) with the maximal value $a = y(0)$ is singular if and only if

$$G(a) \equiv \sqrt{F(a)} \int_0^a \frac{f(a) - f(\tau)}{[F(a) - F(\tau)]^{3/2}}\,d\tau - 2 = 0, \tag{8}$$

where $F(y) = \int\limits_0^y f(t)\,dt$.

2. Nonlinearity as a polynomial of odd degree

Now we study the general case, assuming that the function $f(y)$ is a polynomial, and consequently can change the sign.

We set

$$f(y) = (y - a_1)(y - a_2)(y - a_3)...(y - a_{2n-2})(a_{2n-1} - y), \tag{9}$$

where $0 < a_1 < a_2 < ... < a_{2n-2} < a_{2n-1}$ – isolated zeros of function $f(y)$, i. e. $f(a_i) = 0$. It is obviously that problem (1)–(2) has trivial solutions:

$$y = a_i, \quad i = 1,\, 2,\, \ldots,\, 2n - 1. \tag{10}$$

Here $f(y)$ is a polynomial of odd degree, so it has odd number of zeros. The function (9) is negative on (a_1, a_2), then the function is positive on (a_2, a_3). Therefore, the function has n pairs of humps, where $f(y) > 0$ on (a_{2n-2}, a_{2n-1}) and $f(y) < 0$ on (a_{2n-3}, a_{2n-2}).

We suppose that $f(y)$ satisfies the conditions $F(a_1) < F(a_2) \ldots < F(a_{2n-2}) < F(a_{2n-1})$. Each solution branch has its maximal values inside a single positive hump, and, f. e., that it is necessary to have $\int\limits_{a_1}^{a_3} f(y)\,dy > 0$ in order for solutions with maximal values in (a_2, a_3) to exist. Plots of functions $f(y)$ and $F(y)$ are depicted on Fig. 1.

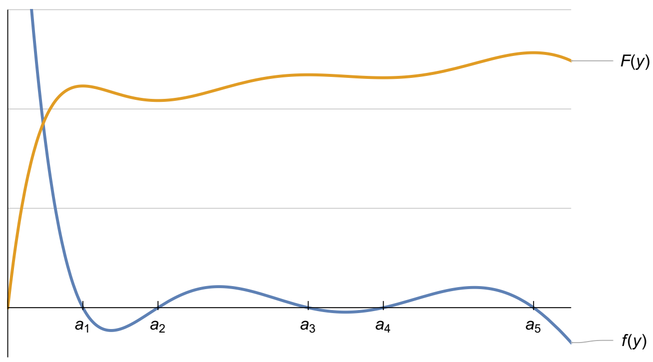


FIGURE 1. $f(y)$ and $F(y)$

Figure 2 shows a plot of the function (8) in the plane $(a; G(a))$, where f corresponds to (9). It follows from the plot that $G(a)$ has zeros only in the intervals (a_2, a_3), (a_4, a_5), $(a_6, a_7), \ldots, (a_{2n-2}, a_{2n-1})$ therefore only in these intervals bifurcation points exist. Also it is clearly that function $G(a)$ exists only on the intervals where $f(y) > 0$.

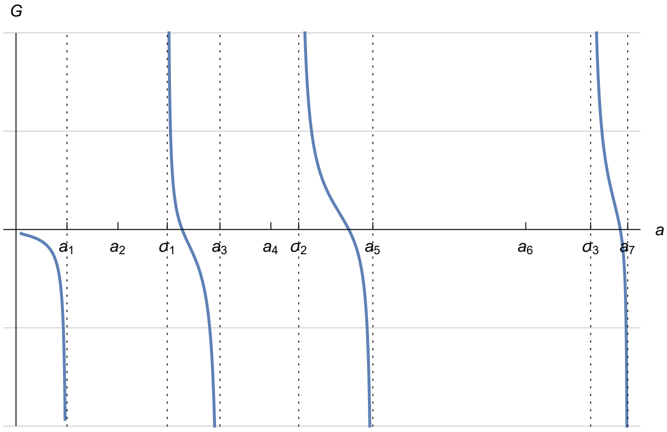


FIGURE 2. $G(a)$ for $f(y)$ - polynomial of odd degree

We will use the asymptotic behavior of $G(a)$ to make the intervals, where $G(a) = 0$, more precise:

1. $\lim\limits_{a \to a_{2n-1}^-} G(a) = -\infty$, i. e., to the left of a_{2n-1} there is no bifurcation point, where $n = 2, \dots, k,\ k \geq 2$.

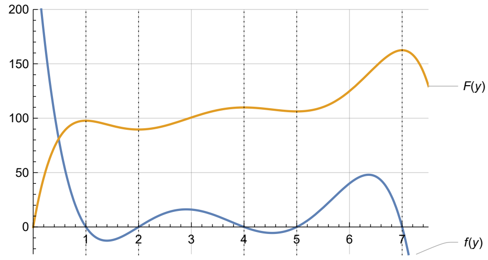
2. Let exist a point $\sigma_{n-1} \in (a_{2n-2}, a_{2n-1})$, such that $\int\limits_{a_{2n-3}}^{\sigma_{n-1}} f(s)\, ds = 0$, so $\lim\limits_{a \to \sigma_{n-1}^+} G(a) = +\infty$, i. e., to the right of σ_{n-1} there is no bifurcation point, where $n = 2, \dots, k$, $k \geq 2$.

Polynomial of third degree (cubic) is well-studied in [4]. Authors show the existence of a critical value of the parameter $\lambda = \lambda_0$, so that for $0 < \lambda < \lambda_0$ the problem (3)–(4) with $f(y) = (y - a_1)(y - a_2)(a_3 - y)$ has exactly one solution, for $\lambda = \lambda_0$ it has exactly two solutions, and exactly three solutions for $\lambda > \lambda_0$.

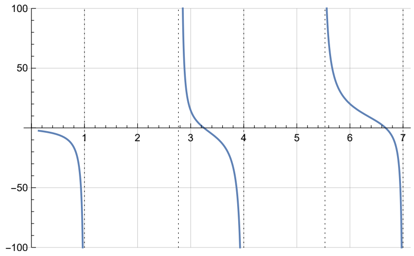
Let consider the examples of polynomials of fifth and seventh degrees.

Example 1. Polynomial of 5th degree.

Let $f(y) = (y-1)(y-2)(y-4)(y-5)(7-y)$ [5]. First, we plot $f(y)$ and $F(y)$ (3(a)) and $G(a)$ (3(b)) to visualize their behavior. It follows from the plot of the function $G(a)$ that there are two bifurcation points on intervals (a_2, a_3), and (a_4, a_5), where $f(y) > 0$ (intervals $(2; 4)$ and $(5; 7)$). We plot a bifurcation diagram as it presented in formula (7) corresponding to this problem (Fig. 4).



(a) $f(y)$ and $F(y)$



(b) $G(a)$

FIGURE 3. Plots of $f(y)$, $F(y)$ and $G(a)$ for example 1

Using accurate commands NMinimize and FindRoot of Wolfram Mathematica we define turning points of $\lambda(a)$: $a_1 \approx 3.2417$, $a_2 \approx 6.5866$, where $\lambda_0 \approx 0.56973$ and $\lambda_1 \approx 0.6321$ (they are sorted in ascending order).

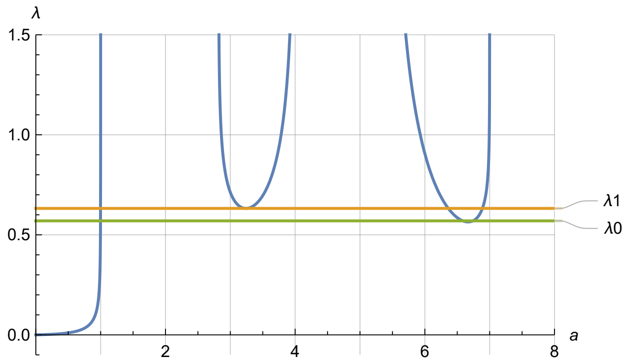
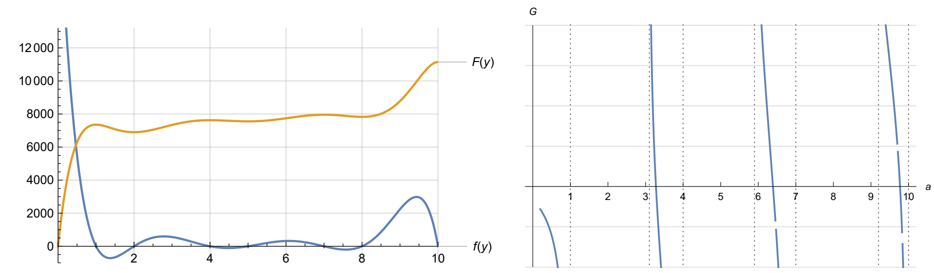


FIGURE 4. $\lambda(a)$ for problem in example 1

As we see on Fig. 4 there exists $0 < \lambda_0 < \lambda_1$ such that for $\lambda < \lambda_0$ there is one solution, $\lambda = \lambda_0$ there are two solutions, for $\lambda_0 < \lambda < \lambda_1$ there are three solutions, for $\lambda = \lambda_1$ there are four solutions and $\lambda > \lambda_1$ there are five solutions to the problem (1)–(2), where $f(y) = (y-1)(y-2)(y-4)(y-5)(7-y)$.

Example 2. Polynomial of 7th degree.
Let $f(y) = (y-1)(y-2)(y-4)(y-5)(y-7)(y-8)(10-y)$. Again we plot $f(y)$ and $F(y)$ (Fig. 5(a)) and $G(a)$ (Fig. 5(b)) to define the intervals where bifurcation points can occur. There are three bifurcation points, each on interval (a_2, a_3), (a_4, a_5) and (a_6, a_7), respectively. These intervals are $(2; 4)$, $(5; 7)$ and $(8; 10)$, where $f(y) > 0$. Bifurcation diagram for this problem is presented at Fig. 6. With the help of numeric computing methods of Wolfram Mathematica we define



(a) $f(y)$ and $F(y)$ (b) $G(a)$

Figure 5. Plots of $f(y)$, $F(y)$ and $G(a)$ for example 2

bifurcation points: $a_1 \approx 3.27276$, $a_2 \approx 6.38791$, $a_3 \approx= 9.6693$, where ordered by ascending values of λ are $\lambda_0 \approx 0,0181$, $\lambda_1 \approx 0.0194$, $\lambda_2 \approx 0.0633$.

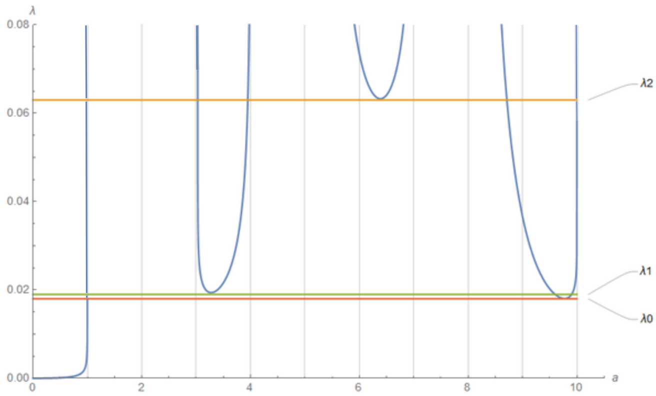


Figure 6. $\lambda(a)$ for problem in example 2

There exist $0 < \lambda_0 < \lambda_1 < \lambda_2$ such that for $\lambda < \lambda_0$ there is one solution to the problem, $\lambda = \lambda_0$ there are two solutions, for $\lambda_0 < \lambda < \lambda_1$ there are three solutions, for $\lambda = \lambda_1$ there are four solutions, $\lambda_1 < \lambda < \lambda_2$ there are five solutions, $\lambda = \lambda_2$ there are six solutions and $\lambda > \lambda_2$ there are seven solutions to the problem.

Conclusion

The obtained results generalize [5]. The study of the function $G(a)$ showed that it has zeros only in the intervals (a_2, a_3), (a_4, a_5), $(a_6, a_7), \ldots, (a_{2n-2}, a_{2n-1})$, where $f(y)$ – polynomial of odd degree ($f(a_i) = 0$), and consequently only these intervals contain bifurcation points. The odd degree of the polynomial $f(y)$ exactly determine the number of solutions of BVP (1)–(2). The bifurcation approach to the problem assists to find out bifurcation parameters λ_i to understand when the number of solutions changes.

Computational methods of numerical integration and differentiation, as well as visualization of $G(a)$ and $\lambda(a)$ in the computing system Wofram Mathematica 11.0, have defined themselves as an effective tool for studying the function $G(a)$ from P. Korman, Y. Li and T. Ouyang Theorem, bifurcation curves and finding out the number of positive solutions of the problem.

References

[1] P. Korman, *Global Solution Branches and Exact Multiplicity of Solutions for Two Point Boundary Value Problems.* Handbook of Differential Equations: Ordinary Differential Equations, 2006.

[2] R. Schaaf, *Global Solution Branches of Two Point Boundary Value Problems*, Lecture Notes in Mathematics, Springer-Verlag, 1990.

[3] P. Korman, Y. Li, T. Ouyang *Computing the location and the direction of bifurcation*, Math. Research Letters, 2005.

[4] P. Korman, Y. Li, T. Ouyang *Exact multiplicity results for boundary value problems with nonlinearities generalising cubic*, Proc. Royal Soc. Edinburgh, 1996.

[5] P. Korman, Y. Li, T. Ouyang *Verification of bifurcation diagrams for polynomial-like equations*, Journal of Computational and Applied Mathematics, 2008.

Daria Chemkaeva
Dep. of Informatics & Technology
Herzen State Pedagogical University
St.Petersburg, Russia
e-mail: dariachemkaeva@yahoo.com

Alexandr Flegontov
Dep. of Informatics & Technology
Herzen State Pedagogical University
St.Petersburg, Russia
e-mail: flegontoff@yandex.ru