Real cubic hypersurfaces containing no line of singular points

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Abstract. We propose a sufficient condition for the absence of a real line consisting of singular points of the given cubic hypersurface over the field of real numbers. There exist examples when this condition is satisfied. There is also an example when this condition is not necessary.

The Bertini theorem provides a simple probabilistic method to check whether the given hypersurface contains a high-dimensional linear subspace of singular points. On the other hand, it is hard to find an isolated singular point of the given cubic hypersurface. Moreover, if the hypersurface as well as all potential singular points are defined over the field of rational numbers, then its singularity recognition is a NP-complete problem.

Let us consider a projective hypersurface $\mathcal{X}$ with a line of singular points. For example, the Whitney umbrella is a ruled surface defined by the form $x_1^2x_3 - x_2^2x_4$; its singular locus contains a line defining by two equations $x_1 = x_2 = 0$.

Let us consider both projective cubic hypersurface $\mathcal{F}$ and hyperplane $\mathcal{H}$ defined by the forms $f(x_0, \dots, x_n)$ and $h(x_0, \dots, x_n)$, respectively. Let a cubic hypersurface $\mathcal{F}_\mathcal{H}$ be defined by the form $h^2x_{n+1} + f$. The hypersurface $\mathcal{F}_\mathcal{H}$ has a singular point with the homogeneous coordinates $(0 : \dots : 0 : 1)$. In the general case, $\mathcal{F}_\mathcal{H}$ is not a cone.

If the hypersurface $\mathcal{F}$ contains a singular point $P \in \mathcal{F} \cap \mathcal{H}$ with the homogeneous coordinates $(p_0 : \dots : p_n)$, then there exists a line of singular points of the hypersurface $\mathcal{F}_\mathcal{H}$; the line contains the point P. Points of this line have coordinates of the type $(p_0 : \dots : p_n : x_{n+1})$, where x_{n+1} is equal to an arbitrary number.

Contrariwise, if a point $\hat{P}$ of the hypersurface $\mathcal{F}_\mathcal{H}$ with the homogeneous coordinates $(p_0, \dots, p_{n+1})$ is singular, and $\hat{P}$ does not coincide with the point $(0 : \dots : 0 : 1)$, then its projection $P \in \mathcal{F}$ with the homogeneous coordinates

$(p_0,\dots,p_n)$ is singular too. Without loss of generality, one can assume that $h = x_n$.

$$\begin{array}{rrcl} \forall k < n & \frac{\partial}{\partial x_k}\left(x_n^2 x_{n+1} + f(x_0,\dots,x_n)\right) & = & \frac{\partial f}{\partial x_k} \\ & \frac{\partial}{\partial x_n}\left(x_n^2 x_{n+1} + f(x_0,\dots,x_n)\right) & = & 2x_n x_{n+1} + \frac{\partial f}{\partial x_n} \\ & \frac{\partial}{\partial x_{n+1}}\left(x_n^2 x_{n+1} + f(x_0,\dots,x_n)\right) & = & x_n^2 \end{array}$$

The last equation implies that $p_n = 0$. Thus, for all coordinates, the equations

$$\frac{\partial f}{\partial x_k}\Big|_P = \frac{\partial}{\partial x_k}\left(x_n^2 x_{n+1} + f(x_0,\dots,x_n)\right)\Big|_{\hat{P}} = 0$$

hold. So, the point P is singular.

In particular, if the hyperplane $\mathcal{H}$ contains no singular point of $\mathcal{F}$, then $\mathcal{F}_{\mathcal{H}}$ has the unique singular point with homogeneous coordinates $(0 : \cdots : 0 : 1)$.

Next, let the cubic hypersurface $\mathcal{F}$ be defined over the field of real numbers. A smooth point $P \in \mathcal{F}$ is said to be elliptic if it is the isolated real point of the intersection of the hypersurface $\mathcal{F}$ and the tangent hyperplane $\mathcal{T}_P$. A sufficient condition for the absence of a real line L consisting of singular points is the existence of an elliptic point $P \in \mathcal{F}$. Let us prove it by contradiction. Assume there exists a real line of singular points. The line intersects all hyperplanes in the projective space. So, the hyperplane section $\mathcal{T}_P \cap \mathcal{F}$ contains both singular points P and $Q \in \mathcal{T}_P \cap L$. Thus, the section contains a real line PQ. But this contradicts the isolation of the real point P.

If a cubic hypersurface contains two real connected components, then the required point exists. The orientable component bounds the convex domain. Therefore, all points of the orientable component are elliptic points. Moreover, if the orientable component of $\mathcal{F}$ does not intersect the hyperplane $\mathcal{H}$, then the hypersurface $\mathcal{F}_{\mathcal{H}}$ contains an elliptic point.

There exists an affine cubic surface having no elliptic point. For example, it is true for the monkey saddle defined by the equation $x_1(x_1^2 - 3x_2^2) - x_3 = 0$.

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