What does a random knot look like?

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Abstract. We discuss the structure and statistics of the set of classical knots and present new results in this research area.

We study the structure and statistical characteristics of the set of classical knots. Closely related topics are statistics of links, tangles, 3-manifolds, graph embeddings, plane diagrams, meanders, countable groups, elements of mapping class groups, braids, etc. The question we primarily address is what does a typical large (prime) knot look like. Probabilistic wording for this question is as follows: what properties does a random (prime) knot have?

What properties of knots does it make sense to check for genericity? Nice candidates come from the basic knot classification. The first level of the classification splits the set of non-trivial knots into the subclasses of prime and composite ones. The second level divides prime knots into the satellite (with incompressible tori in the complement) and simple ones. Similar classifications hold for links, 3-manifolds, etc. Thurston proved that the complement of every simple knot bears a geometric structure: every simple knot is either torus or hyperbolic. This yields the following tree of knots basic properties:

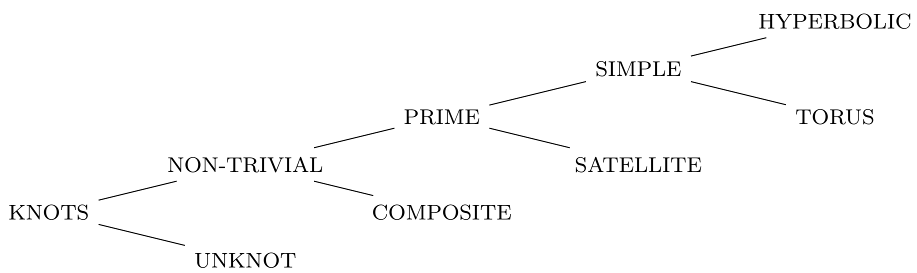

Figure 1. The tree of basic knot classes.

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The dual question to the previous one is which random knot model we choose. Dozens of such models are described in the literature (see [4]). We discuss several of them that are either natural, well-studied, or have specific properties that are of particular interest to our discussion. Our list includes the following:

- random walk models (a random knot is a random polygon in $\mathbb{R}^3$ whose edge vectors are guided by some non-degenerate probability distribution);
- braid group models (we consider knots and links that are Alexander/plat closures of randomly generated braids);
- knot tables model (we consider uniform measures on the sets of knots with crossing number at most n);
- random jump model (a random knot is a random polygon in $\mathbb{R}^3$ whose vertices are guided by the same non-degenerate probability distribution).

An interesting issue related to the above properties of knots is the balance between hyperbolic knots and satellites. For a rather long period of time it was widely believed that most knots and links are hyperbolic (see, e. g., [12, p. 507]). The reason is that the sets of torus and satellite knots look rather special and rare and give an impression of scarcity. In particular, only 32 of the first 1 701 936 prime knots are non-hyperbolic (see [5]). Another related fact is that hyperbolic knots are generic in the braid group models (see [9, 7, 6]).

However, a deeper analysis shows that the conjecture of the hyperbolic knots prevalence is quite flimsy. Indeed, the braid group models are highly imbalanced, and the case with 1 701 936 knots can be explained by the fact that satellites are relatively large, which does not imply asymptotic scarcity. Furthermore, there is (indirect) evidence that the satellites persist in random walk models (see [8, 3]). In addition, the conjecture that hyperbolic knots are asymptotically generic in prime knot tables (see [1]) contradicts several other plausible conjectures (see [10, 11]). This is due to the fact that satellite structures should be large enough, but they can be local (see [11]). We also present the following new evidence related to the tables model.

Theorem 1. *The percentage of hyperbolic links amongst all of the prime non-split links of n or fewer crossings does not tend to* 100 *as n tends to infinity.*

In models where a random knot is satellite, it is interesting to study its companionship tree (see [2]).

It would seem that the above arguments indicate satellite knots predominance. However, we conjecture that the space of knots is complex enough to have several natural well-balanced random knot models showing opposite behavior. In this regard, we present the following new conjecture related to the random jump model.

Conjecture 1. *Hyperbolic knots are generic in the random jump model.*

Expected behavior of distinct random knot models is presented in Table 1.

model \ set of knots	all knots	prime knots
random walk models	composite (proved)	satellite (str. conj.)
braid group models	hyperbolic (proved)	hyperbolic (proved)
knot tables model	composite (str. conj.)	satellite (str. conj.)
random jump model	hyperbolic (weak conj.)	hyperbolic (weak conj.)

TABLE 1. Generic types of knots in several models.

References

[1] C. C. Adams, *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*, New York, W. H. Freeman and Company, 1994.

[2] R. Budney, *JSJ-decompositions of knot and link complements in S^3*, Enseign. Math. (2) **52**:2 (2006), 319–359.

[3] Y. Diao, J. C. Nardo, and Y. Sun, *Global knotting in equilateral random polygons*, J. Knot Theory Ramifications **10**:4 (2001), 597–607.

[4] C. Even-Zohar, *Models of random knots*, J. Appl. Comput. Topology **1**:2 (2017), 263–296.

[5] J. Hoste, M. Thistlethwaite, and J. Weeks, *The first 1,701,936 knots*, Math. Intelligencer **20**:4 (1998), 33–48.

[6] K. Ichihara and J. Ma, *A random link via bridge position is hyperbolic*, Topology Appl. **230** (2017), 131–138.

[7] T. Ito, *On a structure of random open books and closed braids*, Proc. Japan Acad. Ser. A Math. Sci. **91**:10 (2015), 160–162.

[8] D. Jungreis, *Gaussian random polygons are globally knotted*, J. Knot Theory Ramifications **3**:4 (1994), 455–464.

[9] J. Ma, *The closure of a random braid is a hyperbolic link*, Proc. Amer. Math. Soc. **142**:2 (2014), 695–701.

[10] A. V. Malyutin, *Satellite knots strike back*, Abstracts of the International Conference "Polynomial Computer Algebra '15" (2015), 65–66.

[11] A. V. Malyutin, *On the question of genericity of hyperbolic knots*, preprint (2016), arXiv:1612.03368.

[12] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Grad. Texts in Math. 149, Springer, New York, 2nd edition, 2006.

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