On the Extension of Adams–Bashforth–Moulton Methods for Numerical Integration of Delay Differential Equations and Application to the Moon's Orbit

Dan Aksim and Dmitry Pavlov

Laboratory of Ephemeris Astronomy
Institute of Applied Astronomy of the Russian Academy of Sciences
St. Petersburg, Russia

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Types of differential equations

Ordinary differential equation (ODE):

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t))$$

(Retarded) delay differential equation (DDE):

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(\varphi(t)), \ldots),\ \ \varphi(t) < t$$

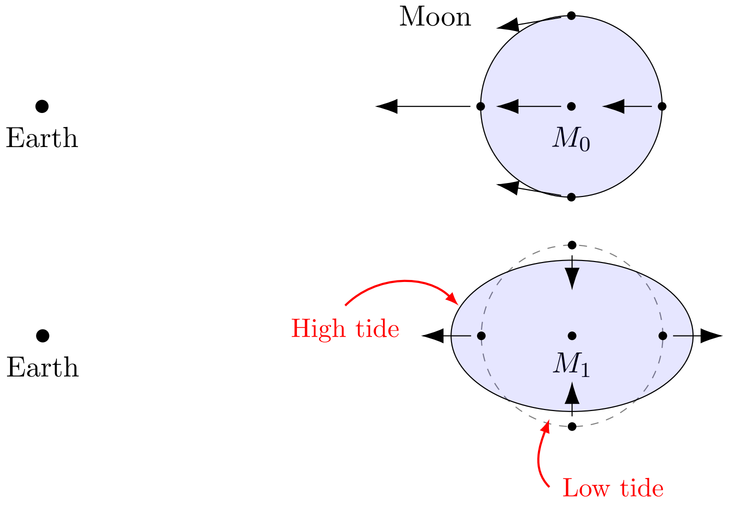
Advanced differential equation:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(\psi(t)), \ldots),\ \ \psi(t) > t$$

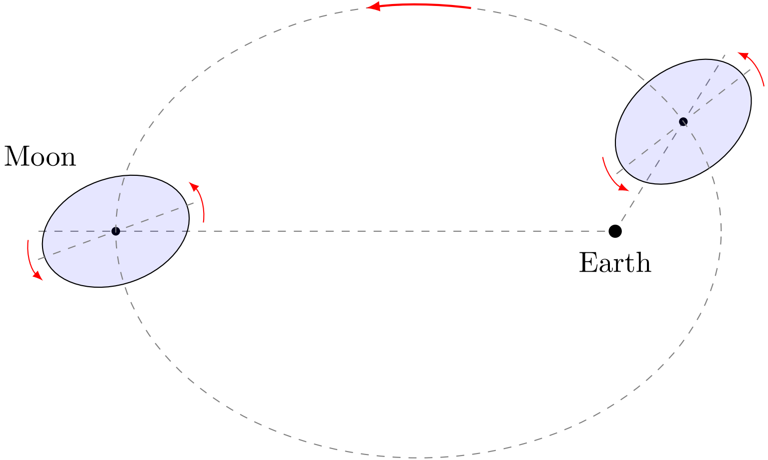
DDE of neutral type:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \dot{\mathbf{x}}(\xi(t)), \ldots),\ \ \xi(t) \neq t$$

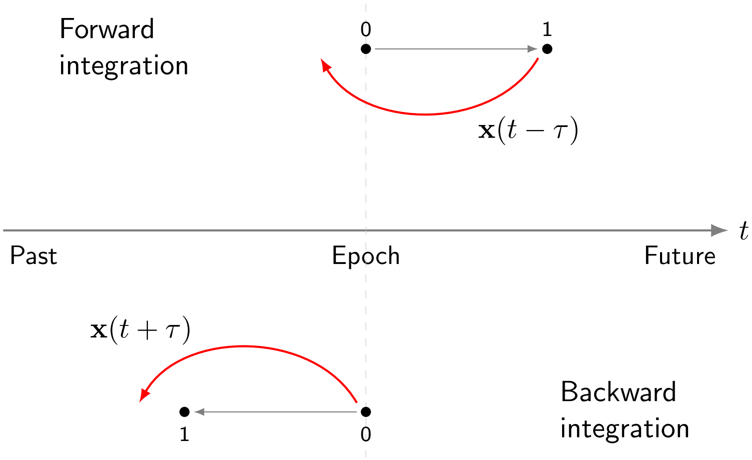
Tidal forces (I)



Tidal forces (II)



From retarded to advanced equations



The Moon equation (general form)

Forward: retarded DDE of neutral type with constant delays

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t-\tau), \dot{\mathbf{x}}(t-\tau))$$

Backward: advanced DDE of neutral type with constant delays

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t+\tau), \dot{\mathbf{x}}(t+\tau))$$

Initial condition at the epoch:

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

The Moon equation (actual form)

Euler's equation for a rotating reference frame:

$$\dot{\boldsymbol{\omega}} = \left(\frac{I}{m}\right)^{-1}\left[\frac{\mathbf{N}}{m} - \frac{\dot{I}}{m}\boldsymbol{\omega} - \boldsymbol{\omega} \times \left(\frac{I}{m}\boldsymbol{\omega}\right)\right],$$

$\boldsymbol{\omega}$ — angular velocity,
$\mathbf{N}(t)$ — torque,
I/m — inertia tensor

$$\frac{I}{m} = \frac{I_0}{m} - \frac{I_c}{m} - k_2\frac{\mu_E}{\mu_M}\left(\frac{R_M}{r}\right)^5\begin{bmatrix} x^2 - \frac{1}{3}r^2 & xy & xz \\ xy & y^2 - \frac{1}{3}r^2 & yz \\ xz & yz & z^2 - \frac{1}{3}r^2 \end{bmatrix}$$
$$+ k_2\frac{R_M^5}{3\mu_M}\begin{bmatrix} \omega_x^2 - \frac{1}{3}(\omega^2 - n^2) & \omega_x\omega_y & \omega_x\omega_z \\ \omega_x\omega_y & \omega_y^2 - \frac{1}{3}(\omega^2 - n^2) & \omega_y\omega_z \\ \omega_x\omega_z & \omega_y\omega_z & \omega_z^2 - \frac{1}{3}(\omega^2 - n^2) \end{bmatrix},$$

where $\mathbf{r} = (x, y, z)^{\mathrm{T}} = \mathbf{r}(t - \tau)$, $\boldsymbol{\omega} = \boldsymbol{\omega}(t - \tau)$, $\tau = 0.096 \text{ d}$

Runge–Kutta methods

General form for the Runge–Kutta family of methods:

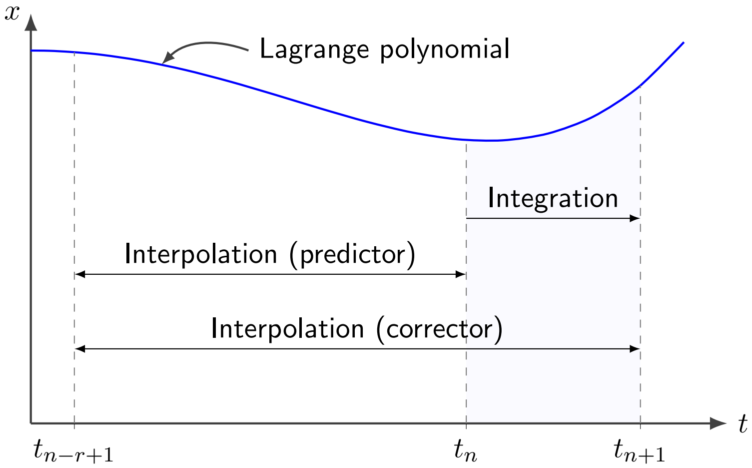
$$\mathbf{x}_{n+1} = \mathbf{x}_n + h \sum_{i=1}^{s} b_i \mathbf{k}_i$$

$$\mathbf{k}_s = f(t_n + c_s h, \mathbf{x}_n + h \sum_{j=1}^{s-1} a_{s,j} \mathbf{k}_j)$$

Drawbacks

- Butcher barriers:
    - $p \geq 5$: no RK method exists of order p with $s = p$ stages
    - $p \geq 7$: no RK method exists of order p with $s = p + 1$ stages
    - $p \geq 8$: no RK method exists of order p with $s = p + 2$ stages
- Higher orders are problematic

Adams–Bashforth–Moulton methods (I)



Adams–Bashforth–Moulton methods (II)

1. Predictor — Adams–Bashforth (order 2):

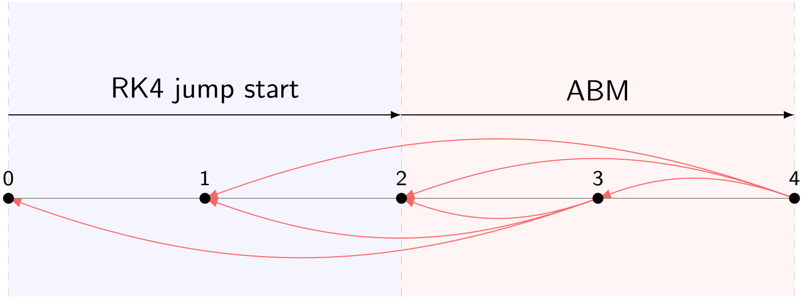
$$\mathbf{x}_{n+2} = \mathbf{x}_{n+1} + h\left(\frac{3}{2}\mathbf{f}_{n+1} - \frac{1}{2}\mathbf{f}_n\right)$$

2. Evaluation of $\mathbf{f}_{n+2} = \mathbf{f}(t_{n+2}, \mathbf{x}_{n+2})$
3. Corrector — Adams–Moulton (order 3):

$$\mathbf{x}_{n+2} = \mathbf{x}_{n+1} + h\left(\frac{5}{12}\mathbf{f}_{n+2} + \frac{2}{3}\mathbf{f}_{n+1} - \frac{1}{12}\mathbf{f}_n\right)$$

4. (Optional). PECE, PECEC, PECECE

Adams–Bashforth–Moulton methods (III)



First $(r-1)$ steps must be performed by a single-step method.

The 'embedded RK4' method for DDEs

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t \pm \tau), \dot{\mathbf{x}}(t \pm \tau))$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

1. Introduce a new function

$$\mathbf{g}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t), \dot{\mathbf{x}}(t))$$

2. Retrieve delayed states by integrating $\mathbf{g}(t)$ with RK4

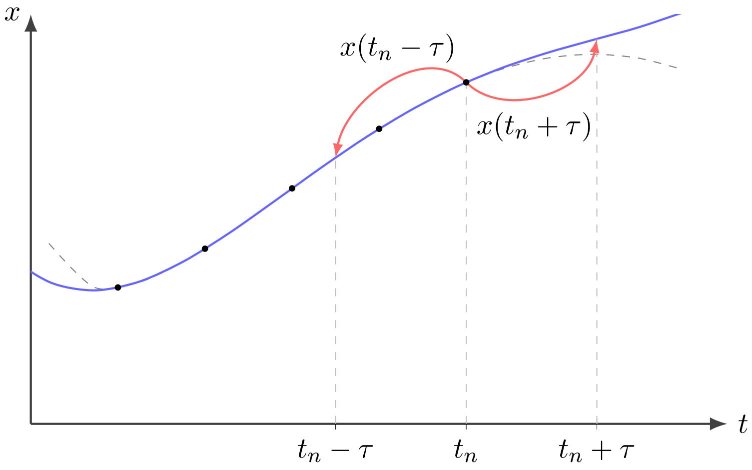
$$\mathbf{x}(t \pm \tau) = {}^{\mathrm{RK4}}\mathcal{I}_{t \to t \pm \tau}\mathbf{g}(t)$$

$$\dot{\mathbf{x}}(t \pm \tau) = \mathbf{g}(t \pm \tau)$$

Drawbacks

- Calculation of each delayed state requires 4 RHS calls
- No previous knowledge of $\mathbf{x}(t)$ is being used

The interpolation method for DDEs (I)



The interpolation method for DDEs (II)

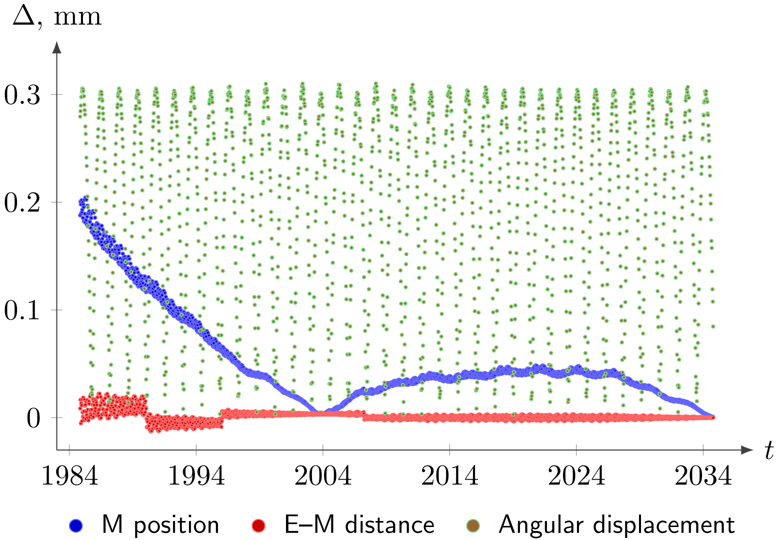
The algorithm

1. Jump start by the 'embedded RK4' algorithm
2. P and C stages are simple ABM
3. Each E stage constructs a Lagrange interpolating polynomial (any order), which is used to find $\mathbf{x}(t \pm \tau)$ and $\dot{\mathbf{x}}(t \pm \tau)$

Advantages

- Much cheaper delayed states
- No simplifying assumptions about the function

Results. The forward–backward test



Orders: ABM — 11, interpolation — 7, extrapolation — 9