Double Hurwitz Numbers

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Abstract. The talk is based on the joint work with N. Do (Monash University). The most straightforward definition of the double Hurwitz numbers $D_g(\mu, \nu)$ is, up to a multiplicative constant, the number of ways to multiply a given permutation of cyclic type μ by a product of $2g - 2 + |\mu| + |\nu|$ transpositions such that the result is of cyclic type ν. It turns out that these number can be packed into generating functions that can be calculated using a recursion. We formulate a conjecture on the analytical properties of these generating functions.

Introduction

The talk is based on the joint work with N. Do (Monash University). Simple Hurwitz numbers enumerate the number of ways to decompose a permutation of a given cyclic type into a product of fixed number of transpositions. Their study was first initiated in nineteenth century by A. Hurwitz. However, they still attract the interest due to incredibly rich structure they possess.

Double Hurwitz numbers are defined in a similar way: we fix two cyclic type μ and ν in the symmetric group S_d and count the number of ways to multiply a permutation of a cyclic type μ by a product of $2g - 2 + |\mu| + |\nu|$ transpositions such that the result is of cyclic type ν.

It is well-known that both simple and double Hurwitz numbers can be interpreted as a number of non-isomorphic ramified covers of $\mathbb{C}P^1$ with certain restriction on the branch points profiles. It allows us to compute double Hurwitz numbers via the enumeration of ramified covers weighted by a certain polynomial weight as follows.

Fix a positive integer d and weights $s, q_1, q_2, \ldots, q_d \in \mathbb{C}$. Define the *double Hurwitz number* $DH_{g,n}(\mu_1, \ldots, \mu_n)$ to be the weighted count of connected genus g branched covers of the Riemann sphere $f : (\Sigma; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- all branching away from 0 and ∞ is simple and occurs at some number m of fixed points;

- $f^{-1}(\infty) = \mu_1 p_1 + \cdots + \mu_n p_n$; and
- no preimage of 0 has ramification index larger than d.

If such a branched cover has ramification profile $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ over 0, then we assign it the weight

$$\frac{q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}}{|\mathrm{Aut}\ f|} \frac{s^m}{m!}.$$

Here, the automorphism group Aut f consists of Riemann surface automorphisms $\phi : \Sigma \to \Sigma$ that preserve the marked points $p_1, \ldots, p_n$ and satisfy $f \circ \phi = f$.

We present an efficient recursion that, in principle, allows to compute all double Hurwitz numbers, and formulate an explicit conjecture concerning the properties of the corresponding generating functions.

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