Schutzenberger transformation on graded graphs: Implementation and numerical experiments.

Vasilii Duzhin and Nikolay Vassiliev

1. Introduction

The Schutzenberger transformation on Young tableaux, also known as "jeu de taquin", was introduced in Schutzenberger's paper [1]. This transformation allows to solve different problems of enumerative combinatorics and representation theory of symmetric groups. Particularly, it can be used to calculate the Littlewood-Richardson coefficients [2].

The connection between Schutzenberger transformation, RSK correspondence [3, 4, 5] and Markov Plancherel process [7] was found in [6]. The techniques discussed in the work [6] have been developed in the recently published paper [8].

We consider the Schutzenberger transformation on two- and three- dimensional Young tableaux. The Schutzenberger transformation converts a Young tableau of size n to another Young tableau of size $n-1$. At the beginning, the first box of a source tableau is being removed. Then, the box with a smaller number is being selected among top neighbouring and right neighbouring boxes. The selected box is then being shifted to the position of the removed box. A newly formed empty box is being filled by the neighbouring box using the same rule. This process continues until the front of the diagram is reached.

The sequence of the shifted boxes forms so-called jeu de taquin path [8] or *Schutzenberger path.* Schutzenberger path is a path in Pascal graphs: $\mathbb{Z}_+^2$ or $\mathbb{Z}_+^3$ in 2D and 3D cases, respectively.

Besides the classic Schutzenberger transformation, in this work we also consider two different modifications of it. In the first modification, we add an extra box in the position of the last shifted box. In this case, the Schutzenberger transformation does not change the shape of a diagram. Also the transformation becomes reversible, i.e. it establishes a bijection on the paths to a diagram. The second modification is a randomization of the classic Schutzenberger transformation. In

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this case a path to a diagram on the third level of Young graph is being selected randomly. The results of numerical experiments suggest that the iterations of the randomized Schutzenberger transformation generate uniform distribution on the paths to a diagram.

A. M. Vershik has noticed that the Schutzenberger algorithm can be applied not only to the Young tableaux of an arbitrary dimension, but generally to any partially ordered set. In this case the Schutzenberger transformation works on ascendant sequences of decreasing ideals of a corresponding poset. Particularly, the technique of the Schutzenberger transformation can be used on any graded graph. In this situation, a Schutzenberger path will be a path on this graded graph.

It was proved in [8] that the Schutzenberger paths, obtained on two-dimensional Young tableaux, have a certain limit angle with a probability 1 relatively to the Plancherel measure.

Note that the standard Schutzenberger transformation is not reversible. Wherein each Young tableau has as many preimages as the number of transitions from a given diagram of size n to the level $n+1$. Fig. 1 shows the Schutzenberger paths of all preimages of the 2D Young tableau of size 10^6.

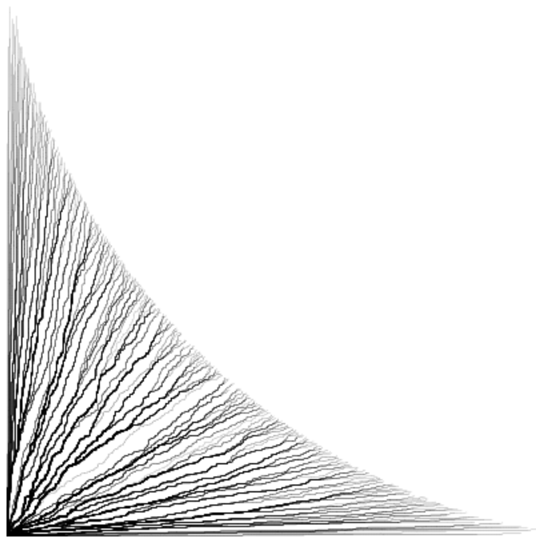

FIGURE 1. The Schutzenberger paths of all preimages of the Young tableau of size 10^6.

1.1. The implementation of the Schutzenberger transformation

We propose the following algorithm for the implementation of the Schutzenberger transformation on 2D and 3D Young tableaux. The same algorithm with minor modifications can be applied to any graded graph. We present Young tableaux as arrays of sets of coordinates of added boxes. Note that the standard presentation of Young tableaux as two-dimensional arrays of integers has a significant disadvantage with respect to the computational cost and memory usage. That is because in that case the Schutzenberger transformation requires renumbering of all boxes in a tableau.

Let us consider the implemented algorithm for the case of 2D Young tableaux. During operation of the algorithm, the coordinates of boxes of a source tableau are processed consequently. At the beginning, the first box of a source tableau with coordinates (0,0) is assigned as an *active* box. However, the active box is not being added to a new tableau immediately.

The active box is being added to a new tableau at the moment when a neighbour top or neighbour right box is added to a source tableau. As a next step, this neighbour box becomes active and so on. At the same time, the non-neighbour boxes are being added without any delay. The algorithm stops when all the boxes in a source tableau are processed.

Note that during operation of the algorithm, most of the boxes of a source table are being copied to a new tableau without any changes. Only the order of addings of active boxes will be different. Another advantage of this approach is that after necessary modifications it can be easily implemented on a Young graph of any dimension and on any other graded graphs.

We use the same methods to implement the modifications of Schutzenberger transformation, i.e. the Schutzenberger transformation with the preservation of shape of a diagram and with randomization. The fragment of the algorithm of Schutzenberger transformation in 2D case, written in pseudocode, is shown below. Note that $actX, actY$ are the coordinates of the current active box, *in_tab* is a source tableau and *out_tab* is a transformed tableau.

Listing 1. Schutzenberger transformation on 2D Young tableaux

```
actX = 0; actY = 0;
for each (x,y) from in_tab:
{
    if ((x == actX + 1) && (y == actY)) ||
       ((x == actX)     && (y == actY + 1))
    {
        out_tab.add(actX,actY);
        actX = x; actY = y;
    }
    else
    {
        out_tab.add(x,y);
    }
}
out_tab.add(actX,actY);
```

2. Numerical experiments

Here we discuss the numerical experiments where the Schutzenberger transformation was applied on large 2D and 3D Young tableaux. Particularly, we have generated a 2D random Plancherel Young tableau of 3 million boxes. The Schutzenberger transformation with the preservation of shape was consequently applied to this tableau. The Vershik-Kerov coordinates $\frac{x+y}{\sqrt{3 \cdot 10^6}}$ of the last boxes (x,y) of Schutzenberger paths were recorded. The distribution of these coordinates is shown in Fig. 2.

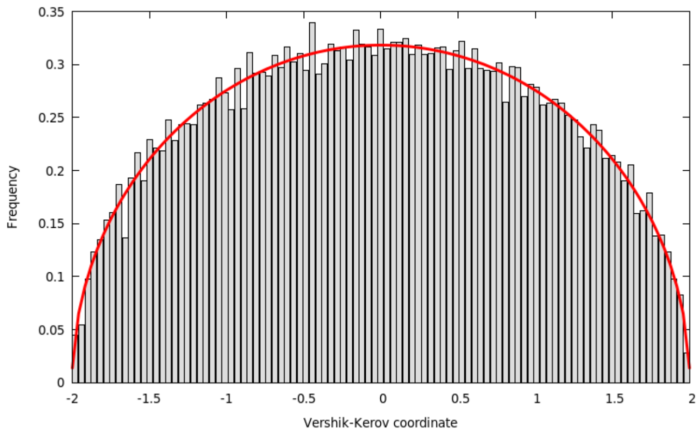

FIGURE 2. The histogram of frequencies of last boxes of Schutzenberger paths of 2D Young tableaux.

It can be seen from the figure that this histogram has the shape of so-called semicircle distribution. The same distribution was obtained in [9] as a limit distribution of Plancherel probabilities on the front of large Young diagrams of size $n, n \to \infty$. It has the following density function:

$$d\mu(u) = \frac{\sqrt{4 - u^2}}{2 \cdot \pi},$$

where u is one of Vershik-Kerov coordinates: $u = \frac{x-y}{\sqrt{n}}$.

The next numerical experiment is devoted to the Schutzenberger transformation on 3D Young graph. For each iteration of the Schutzenberger transformation we compute the coordinates of the last boxes of Schutzenberger paths on the front of random Young tableaux of a fixed shape. The distribution of the coordinates obtained in this experiment is shown in Fig. 3. Note that the size of the corresponding tableau is 3 million boxes.

As we can see, the distribution of last boxes of 3D Schutzenberger paths on the front of the diagram is close to uniform. We plan to conduct more numerical experiments to investigate this 3D distribution more precisely.

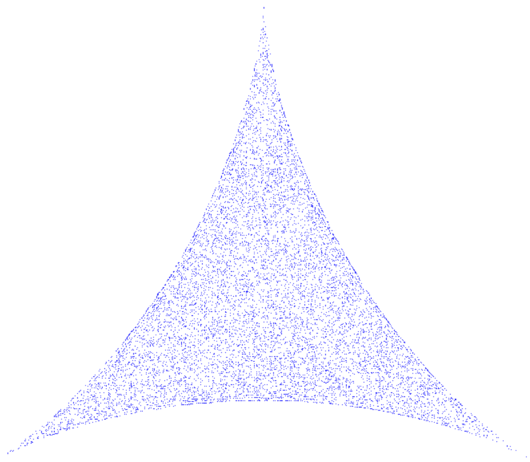

FIGURE 3. The distribution of coordinates of last boxes of Schutzenberger paths of 3D Young tableaux.

Also we used the randomized Schutzenberger transformation to calculate the ratio of dimensions of a pair of three-dimensional Young diagrams of sizes n and $n+1$ which differ in a single box, i. e. a pair of diagrams connected with an edge in the Young graph. The co-transition probabilities of 3D central processes can be obtained using such ratios. The Schutzenberger transformation gives us these co-transition probabilities without calculating the exact dimensions. That is especially useful because there are no known three-dimensional analog of the 2D hook length formula.

References

[1] M. P. Schützenberger, "Quelques remarques sur une construction de Schensted", *Math. Scandinavica* 12, (1963), 117-128.

[2] S.V.Fomin, Knuth equivalence, jeu de taquin, and the Littlewood-Richardson rule, Appendix 1 to Chapter 7 in: R.P.Stanley, Enumerative Combinatorics, vol 2, Cambridge University Press.

[3] G. de B. Robinson, "On the representations of the symmetric group", *American Journal of Math.* 60, (1938), 745-760.

[4] C. Schensted, "Longest increasing and decreasing subsequences", *Canadian Journal of Math.* 13, (1961), 179-191.

[5] Donald E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.* Volume 34, Number 3 (1970), pp. 709-727.

[6] Sergei V. Kerov and Anatol M. Vershik. The characters of the infinite symmetric group and probability properties of the Robinson-Schensted-Knuth algorithm. *SIAM J. Algebraic Discrete Methods*, 7(1):116–124, 1986

[7] S.V.Kerov and A.M.Vershik. Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux. *Dokl. Akad. Nauk SSSR* 233, No.6, 1024-1027 (1977).
English translation: Sov. Math. Dokl. 18, 527-531 (1977).

[8] Dan Romik and Piotr Sniady. Jeu de taquin dynamics on infinite Young tableaux and second class particles. *Annals of Probability: An Official Journal of the Institute of Mathematical Statistics*, 43(2):682-737, 2015

[9] S. V. Kerov, "Transition Probabilities for Continual Young Diagrams and the Markov Moment Problem", *Funktsional. Anal. i Prilozhen.*, 27:2 (1993), 32–49; *Funct. Anal. Appl.*, 27:2 (1993), 104–117

Vasilii Duzhin
Saint Petersburg Electrotechnical University
ul. Professora Popova 5, 197376 St. Petersburg, Russian Federation
e-mail: vduzhin.science@gmail.com

Nikolay Vassiliev
St.Petersburg department of Steklov Institute of mathematics RAS
nab.Fontanki 27,St.Petersburg, 191023, Russia
e-mail: vasiliev@pdmi.ras.ru