# On the Cayley-Bacharach Property

Martin Kreuzer, Le Ngoc Long and Lorenzo Robbiano

**Abstract.** The Cayley-Bacharach property, which has been classically stated as a property of a finite set of points in an affine or projective space, is extended to arbitrary 0-dimensional affine algebras over arbitrary base fields. We present characterizations and explicit algorithms for checking the Cayley-Bacharach property directly, via the canonical module, and in combination with the property of being a locally Gorenstein ring. Moreover, we characterize strict Gorenstein rings by the Cayley-Bacharach property and the symmetry of their Hilbert function, as well as by the strict Cayley-Bacharach property and the last difference of their Hilbert function.

## Extended Abstract

The Cayley-Bacharach Property (CBP) has a long and rich history. Classically, it has been formulated geometrically as follows: *A set of points $\mathbb{X}$ in $n$-dimensional affine or projective space is said to have the Cayley-Bacharach property of degree $d$ if any hypersurface of degree $d$ which contains all points of $\mathbb{X}$ but one automatically contains the last point.* After a brief recap of its history, we present the currently most general version, namely the definition first given in Ngoc Le Long's Thesis (University of Passau, 2015). Our goal is to study this very general version of the CBP and to find efficient algorithms for checking it. A special emphasis is given to algorithms which will us to apply them to families of 0-dimensional ideals parametrized by border basis schemes. Moreover, we generalize the main results about the CBP of many previous papers to this most general setting of a 0-dimensional affine algebra over an arbitrary base field.

To achieve these goals, we proceed as follows. Our main object of study is a 0-dimensional affine algebra $R = P/I$ over an arbitrary field $K$, where we let $P = K[x_1, \ldots, x_n]$ be a polynomial ring over $K$ and $I$ a 0-dimensional ideal in $P$. Even if we do not specify it explicitly everywhere, we always consider $R$ together with this fixed presentation. In other words, we consider a fixed 0-dimensional subscheme $\mathbb{X} = \operatorname{Spec}(P/I)$ of $\mathbb{A}^n$.

This corresponds to the classical setup. However, in the last decades it has been customary to consider 0-dimensional subschemes of projective spaces. Of course, via the standard embedding $\mathbb{A}^n \cong D_+(x_0) \subset \mathbb{P}^n$, the classical setup can be translated to this setting in a straightforward way. For instance, in this case the affine coordinate ring $R = K[x_1, \dots, x_n]/I$ has to substituted by the homogeneous coordinate ring $R^{\rm hom} = K[x_0, \dots, x_n]/I^{\rm hom}$, etc. In this talk we use the affine setting for several reasons: firstly, the ideals defining subschemes of $\mathbb{X}$ can be studied using the decomposition into local rings, secondly, the structure of the coordinate ring of $\mathbb{X}$ and its canonical module can be described via multiplication matrices, and thirdly, the affine setup is suitable for generalizing everything to families of 0-dimensional ideals via the border basis scheme.

First we recall the primary decomposition $I = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_s$ of $I$, the corresponding primary decomposition $\langle 0 \rangle = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ of the zero ideal of $R$, and the decomposition $R = R/\mathfrak{q}_1 \times \cdots \times R/\mathfrak{q}_s$ of $R$ into local rings. Then, for $i \in \{1, \dots, s\}$, a minimal $\mathfrak{Q}_i$-divisor $J$ of $I$ is defined in such a way that the corresponding subscheme of $\mathbb{X}$ differs from $\mathbb{X}$ only at the point $p_i = \mathcal{Z}(\mathfrak{M}_i)$ and has the minimal possible colength $\ell_i = \dim_K(P/\mathfrak{M}_i)$, where $\mathfrak{M}_i = \mathrm{Rad}(\mathfrak{Q}_i)$. In the reduced case, these subschemes are precisely the sets $\mathbb{X} \setminus \{p_i\}$ appearing in the classical formulation of the Cayley-Bacharach Theorem.

Moreover, in order to have a suitable version of degrees, we recall the degree filtration of $R$, its affine Hilbert function $\mathrm{HF}^a_R$, and its regularity index $\mathrm{ri}(R)$. Here the affine Hilbert function plays the role of the usual Hilbert function if we consider affine algebras such as $R$.

These constructions are combined with the definition and some characterizations of *separators*. Then we show that a separator for a maximal ideal $\mathfrak{m}_i$ of $R$ corresponds to a generator of a minimal $\mathfrak{Q}_i$-divisor $J$ of $I$, and we use the maximal order of such a separator to describe the regularity index of $J/I$. Then the minimum of all regularity indices $\mathrm{ri}(J/I)$ is called the *separator degree* of $\mathfrak{m}_i$. We go on to show that this "minimum of all maxima" definition is the correct, but rather subtle generalization of the classical notion of the least degree of a hypersurface containing all points of $\mathbb{X}$ but $p_i$.

The separator degree of a maximal ideal $\mathfrak{m}_i$ of $R$ is bounded by the regularity index $\mathrm{ri}(R)$, since the order of any separator is bounded by this number. If all separator degrees attain this maximum value, we say that $R$ has the *Cayley-Bacharach property (CBP)*, or that $\mathbb{X}$ is a *Cayley-Bacharach scheme*. At this point we construct our first new algorithm which allows us to check whether a given maximal ideal $\mathfrak{m}_i$ of $R$ has maximal separator degree.

Although this algorithm can be used to check the CBP of $R$, we then construct a better one based on the canonical module $\omega_R = \mathrm{Hom}_K(R, K)$ of $R$. The module structure of $\omega_R$ is given by $(f\,\varphi)(g) = \varphi(fg)$ for all $f, g \in R$ and all $\varphi \in \omega_R$. It carries a degree filtration $\mathcal{G} = (G_i\omega_R)_{i \in \mathbb{Z}}$ which is given by $G_i\omega_R = \{\varphi \in \omega_R \mid \varphi(F_{-i-1}R) = 0\}$ and its affine Hilbert function which satisfies $\mathrm{HF}^a_{\omega_R}(i) = \dim_K(R) - \mathrm{HF}^a_R(-i-1)$ for $i \in \mathbb{Z}$. Generalizing some earlier results, we show that the module structure of $\omega_R$ is connected to the CBP of $R$. More precisely, one main theorem says that $R$ has the CBP if and only if $\mathrm{Ann}_R(G_{-\mathrm{ri}(R)}\omega_R) = \{0\}$. Based on this characterization and the description of the structure of $R$ and the module structure of $\omega_R$ via multiplication matrices, we obtain the second main algorithm for checking the CBP of $R$ using the canonical module. As a nice and useful by-product, we show that, for an extension field $L$ of $K$, the ring $R$ has the CBP if and only if $R \otimes_K L$ has the CBP.

Next we turn our attention to 0-dimensional affine algebras $R$ which are locally Gorenstein and have the CBP. We show that $R$ is locally Gorenstein if and only if $\omega_R$ contains an element $\varphi$ such that $\mathrm{Ann}_R(\varphi) = \{0\}$ and that we can check this effectively. Then we characterize locally Gorenstein rings having the CBP by the existence of an element $\varphi \in \omega_{R\otimes L}$ of order $-\mathrm{ri}(R)$ with $\mathrm{Ann}_{R\otimes L}(\varphi) = \{0\}$. Here we may have to use a base field extension $K \subseteq L$ or assume that $K$ is infinite. This characterization implies useful inequalities for the affine Hilbert function of $R$ and allows us to formulate an algorithm which checks whether $R$ is a locally Gorenstein ring having the CBP using the multiplication matrices of $R$. To end this discussion, we characterize the CBP of $R$ in the case when the last difference $\Delta_R = \mathrm{HF}_R(\mathrm{ri}(R)) - \mathrm{HF}_R(\mathrm{ri}(R) - 1)$ is one.

The subsequent topic is to characterize 0-dimensional affine algebras which are strict Gorenstein rings. This property means that the graded ring $\mathrm{gr}_{\mathcal{F}}(R)$ with respect to the degree filtration is a Gorenstein ring. In the projective case, the corresponding 0-dimensional schemes are commonly called arithmetically Gorenstein. Our first characterization of strict Gorenstein rings improves earlier results by Davis, Geramita, and Orecchia. More precisely, we show that $R$ is strictly Gorenstein if and only if it has the CBP and a symmetric Hilbert function. In particular, it follows that these rings are locally Gorenstein. Then we define the strict CBP of $R$ by the CBP of $\mathrm{gr}_{\mathcal{F}}(R)$ and show that it implies the CBP of $R$. Thus we obtain a second characterization of strict Gorenstein rings: $R$ is a strict Gorenstein ring if and only if $R$ has the strict CBP and $\Delta_R = 1$.

In the last part of the talk, we show how one can extend all these characterizations to families of 0-dimensional polynomial ideals. More precisely, we introduce the border basis scheme and explain some ways of getting explicit polynomial equations defining subschemes corresponding to all ideals with a particular property, for instance the CBP.

Martin Kreuzer
Fakultät für Informatik und Mathematik, Universität Passau, D-94030 Passau, Germany
e-mail: Martin.Kreuzer@uni-passau.de

Le Ngoc Long
Fakultät für Informatik und Mathematik, Universität Passau, D-94030 Passau, Germany
e-mail: nglong16633@gmail.com

Lorenzo Robbiano
Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, I-16146 Genova, Italy
e-mail: lorobbiano@gmail.com