Elliptic functions and finite difference method

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Abstract. For some autonomous dynamic systems, especially for equations related to pendulum oscillation and top rotation, new finite difference schemes are suggested.

Standard considerations of integrability of differential equations in symbolic form refer to the computational techniques of the past centuries. For example, in the studies of Galois differential theory, they use the concept of elementary functions. In the times of Liouville, when these studies were initiated, the functions have been considered elementary if their tables were available for common use. At present, this class of functions is much narrower than the set of functions, for which the computation algorithms are implemented in all systems of computer algebra.

Among different approaches to the concept of integrability in the symbolic form a special position is occupied by the approach proposed in the first papers by Painlevé and then forgotten for long [1]. The idea is that in most cases, when the differential equation

$$F(x, y, y', \dots) = 0$$

is integrated in a finite form, the general solution depends on the integration constants algebraically, even when the dependence upon x is described by rather complicated transcendental functions. This property is purely algebraic, which makes it possible to construct an analogue of the Galois theory, where the class of admissible transcendental operations is described in the course of the theory development, rather than being fixed as in the Liouville approach. The approach yields all classical transcendental function, from cylindrical to Abelian ones [2, 3].

There is a rather nontrivial connection between this version of Galois theory, and the finite difference method. If the general solution of the differential equation

$$y' = f(x, y), \quad f \in \mathbb{Q}(x, y),$$

depends on the constant algebraically, then it can be approximated by a difference scheme, defining the projective correspondence between the layers. The main difference of this scheme from the commonly used ones is that it allows one to continue the calculation beyond the movable singular points without considerable accumulation of error [4].

A natural assumption is that for other classes of sets of differential equations, for which the general solution depends on the constants algebraically, one can construct difference schemes that allows one to continue the computations over wider intervals. In other words, the classical transcendental functions are particular solutions of such differential equations, for which the solution using the finite difference method is particularly efficient. In the present report, we would like to consider one of the most important class of such functions, namely, the elliptic functions.

The elliptic functions appeared in mechanics as higher transcendental functions that provide the integration of various problems related to pendulum oscillation and top rotation. In these cases, the equations of motion can be considered as particular cases of the autonomous dynamic system

$$\dot{\vec{x}} = f(\vec{x}),$$

possessing a few integrals of motion. Standard explicit difference schemes do not conserve these integrals; therefore, their use for calculations over the time of about ten periods yields an approximate solution that is considerably aperiodic. For a top with the fixed center of gravity, the integrals of motion are quadratic, so they are conserved by the symplectic Runge-Kutta schemes [5]. However, in the present case all these schemes will be implicit. For example, the transition from one layer to another in the single-stage scheme will require the solution of a fifth-power equation, which essentially complicates practical application of such schemes.

Problem. Given a system of differential equations and a few integrals, construct an explicit difference scheme, exactly conserving the integrals of motion.

In the present report we restrict ourselves to the case, when the integrals of motion specify a curve of the genus ρ in the space, where $\vec{x}$ varies. In this case, any difference scheme is an algebraic transformation of this curve. Therefore, at $\rho > 1$ the desired difference scheme does not exist for purely geometric reasons (Zeuthen-Hurwitz theorems). At $\rho = 1$ (elliptic curve) the explicit formulae for birational transformations are known, so that it is always possible to check, whether these formulae approximate the given differential equation or not. In the case of a top, the answer is positive, and an explicit difference scheme for the approximate solutions can be constructed. Without any relation to the theory of difference schemes, this method of calculating the elliptic functions has been applied to the composition of the first tables by Ch. Gudermann [6]. In the report, we consider the properties of this scheme.

We believe that all transcendental functions can be reconsidered as solutions of such differential equations, for which the application of the finite difference method is particularly efficient.

References

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