Irreducible Decomposition of Representations of Finite Groups via Polynomial Computer Algebra

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Vladimir Kornyak

Laboratory of Information Technologies
Joint Institute for Nuclear Research
Dubna, Russia

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Motivation: decomposition of a quantum system into subsystems

- Quantum system
  - States: Hilbert space $\mathcal{H}$
  - Evolution: unitary operators $\mathsf{U}(g)$ on $\mathcal{H}$, $g \in \mathsf{G}$ – symmetry group
  - Observation: orthogonal projection into subspace $\mathcal{H}$
    measurement = observation in eigensubspaces of Hermitian operator A called "observable"

- Decomposition kinds
  1. tensor product: $\mathcal{H} = \bigotimes_x \mathcal{H}_x,\ x \in X$ – "spatially separated subsystems"
     idea of locality is an approximation due to quantum entanglement
  2. subrepresentations of G: $\mathcal{H} = \bigoplus_m \mathcal{H}_m$ – invariant subspaces
     independent quantum subsystem in each subspace

Permutation group

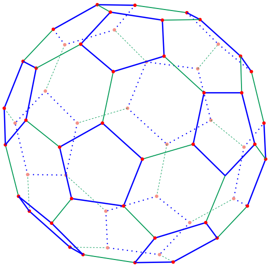
$G(\Omega)$ is a group of bijections of a set $\Omega \cong \{1, \ldots, N\}$

i^g denotes action of $g \in G$ on $i \in \Omega$

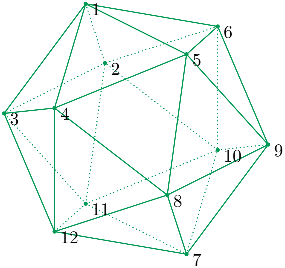
Cayley's theorem: any axiomatic group is a permutation group

Example

$G = A_5$



Ω = vertices of icosahedron $\cong \{1, \ldots, 12\}$



$A_5 = \left\langle a, b \,\middle|\, a^5 = b^2 = (ab)^3 = \mathbf{1} \right\rangle$

$S = \{a, b\} = \{(1, 2, 11, 12, 4)(5, 6, 10, 7, 8),\ (1, 7)(2, 8)(3, 12)(4, 11)(5, 10)(6, 9)\}$

$S = \{s_1, \ldots, s_K\} \subseteq G$ is generating set: $G = \langle S \rangle$

$s_1, \ldots, s_K$ are generators

Permutation representation

is representation G in N–dimensional vector space over field $\mathcal{F}$
by matrices $\mathrm{P}(g)_{ij} = \delta_{ig\,j}$
$\mathrm{P}(g)$ is $(0,1)$-matrix $\Longrightarrow \mathcal{F}$ can be any field
any representation is a subrepresentation of some permutation representation

1. $\mathcal{F}$ is a finite field $\mathrm{GF}(q)$
   - *MeatAxe* is an efficient algorithm of Las Vegas type
     "matrix groups, having degrees up to the high hundreds"
     important role in classification of finite simple groups
     inefficient in characteristic 0
   - Hilbert spaces over $\mathrm{GF}(q)$ are problematic $\Longrightarrow$ little use for physics
2. $\mathcal{F}$ is a constructive field of characteristic 0
   - $\mathcal{F}$ contains a minimal splitting field for G
   - $\mathcal{F}$ is an abelian extension of $\mathbb{Q}$ $\Longrightarrow$
     $\mathcal{F}$ is constructive dense subfield of $\mathbb{R}$ or $\mathbb{C}$

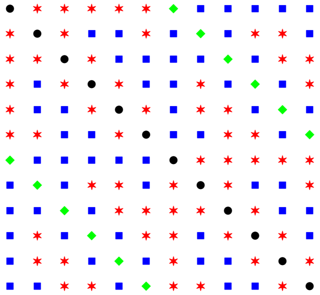
Orbitals and suborbits

1. Orbital (coherent Schur configuration): orbit of G on $\Omega \times \Omega$
2. Rank R of G on Ω is the number of orbitals $\Delta_r \in \{\Delta_1, \ldots, \Delta_{\mathrm{R}}\}$
3. Stabilizer of $i \in \Omega$: $G_i = \{g \in G \mid i^g = i\} \leqslant G$
4. Suborbit $=$ orbit of stabilizer

   Natural one-to-one correspondence between orbitals and suborbits

   $\Delta \longleftrightarrow \Sigma_i = \{j \in \Omega \mid (i, j) \in \Delta\}$

Example: A_5 on icosahedron



Invariant bilinear forms and centralizer ring

1. Invariant bilinear form:

$$A = \mathrm{P}(g) A\, \mathrm{P}\left(g^{-1}\right) \Longleftrightarrow (A)_{ij} = (A)_{i^g j^g} \qquad g \in \mathsf{G}$$

2. Basis of invariant forms: matrices $\{\mathcal{A}_1, \ldots, \mathcal{A}_{\mathrm{R}}\}$
   One-to-one correspondence with orbitals:

$$(\mathcal{A}_r)_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \Delta_r \\ 0, & \text{if } (i,j) \notin \Delta_r \end{cases}$$ — characteristic function of Δ_r on $\Omega \times \Omega$

   Always assume $\mathcal{A}_1 = \mathbb{1}_{\mathsf{N}}$

3. Centralizer ring:

$$\mathcal{A}_p \mathcal{A}_q = \sum_{r=1}^{\mathrm{R}} C_{pq}^r \mathcal{A}_r \qquad C_{pq}^r \in \mathbb{N} = \{0, 1, \ldots\}$$

$$\left\{\begin{array}{l}\text{centralizer ring}\\ \text{is commutative}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\text{representation } \mathrm{P}\\ \text{is multiplicity-free}\end{array}\right\}$$

A_5 on icosahedron: basis of centralizer ring

$\Omega \times \Omega = \{1, \ldots, 12\} \times \{1, \ldots, 12\}$ Rank $\mathrm{R} = 4$

$$\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 = \begin{pmatrix}
\bullet & \ast & \ast & \ast & \ast & \ast & \blacklozenge & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\ast & \bullet & \ast & \blacksquare & \blacksquare & \ast & \blacksquare & \blacklozenge & \blacksquare & \ast & \ast & \blacksquare \\
\ast & \ast & \bullet & \ast & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacklozenge & \blacksquare & \ast & \ast \\
\ast & \blacksquare & \ast & \bullet & \ast & \blacksquare & \blacksquare & \ast & \blacksquare & \blacklozenge & \blacksquare & \ast \\
\ast & \blacksquare & \blacksquare & \ast & \bullet & \ast & \blacksquare & \ast & \ast & \blacksquare & \blacklozenge & \blacksquare \\
\ast & \ast & \blacksquare & \blacksquare & \ast & \bullet & \blacksquare & \blacksquare & \ast & \ast & \blacksquare & \blacklozenge \\
\blacklozenge & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \bullet & \ast & \ast & \ast & \ast & \ast \\
\blacksquare & \blacklozenge & \blacksquare & \ast & \ast & \blacksquare & \ast & \bullet & \ast & \blacksquare & \blacksquare & \ast \\
\blacksquare & \blacksquare & \blacklozenge & \blacksquare & \ast & \ast & \ast & \ast & \bullet & \ast & \blacksquare & \blacksquare \\
\blacksquare & \ast & \blacksquare & \blacklozenge & \blacksquare & \ast & \ast & \blacksquare & \ast & \bullet & \ast & \blacksquare \\
\blacksquare & \ast & \ast & \blacksquare & \blacklozenge & \blacksquare & \ast & \blacksquare & \blacksquare & \ast & \bullet & \ast \\
\blacksquare & \blacksquare & \ast & \ast & \blacksquare & \blacklozenge & \ast & \ast & \blacksquare & \blacksquare & \ast & \bullet
\end{pmatrix}$$

Splitting invariant inner product

1. T is unitary transformation splitting P into M irreducible components:
$$T^{-1}\mathrm{P}(g)\,T = 1 \oplus \mathsf{U}_{d_2}(g) \oplus \cdots \oplus \mathsf{U}_{d_m}(g) \oplus \cdots \oplus \mathsf{U}_{d_M}(g)$$
2. $\mathbb{1}_\mathrm{N}$ is standard inner product in any orthonormal basis
3. Decomposition of $\mathbb{1}_\mathrm{N}$ in splitting basis
$$\mathbb{1}_\mathrm{N} = \mathbb{1}_{d_1=1} \oplus \cdots \oplus \mathbb{1}_{d_m} \oplus \cdots \oplus \mathbb{1}_{d_M}$$
4. Inverse image of the decomposition in original permutation basis
$$\mathbb{1}_\mathrm{N} = \mathcal{B}_1 + \cdots + \mathcal{B}_m + \cdots + \mathcal{B}_M$$
5. $\mathcal{B}_m$ is inverse image of irreducible component:
$$T^{-1}\mathcal{B}_m T = \mathbb{0}_{1+d_2+\cdots+d_{m-1}} \oplus \mathbb{1}_{d_m} \oplus \mathbb{0}_{d_{m+1}+\cdots+d_M} \equiv \mathcal{D}_m$$
   - $\mathcal{B}_m$ is projector: $\mathcal{B}_m^2 = \mathcal{B}_m$ $\quad \mathcal{B}_m\mathcal{B}_{m'} = 0$ if $m \neq m'$
   - $\operatorname{tr}\mathcal{B}_m = d_m$

$\mathcal{B}_m$'s carry full information about irreducible decomposition of P

Linear system for T: $\mathcal{B}_1 T - T\mathcal{D}_1 = \cdots = \mathcal{B}_M T - T\mathcal{D}_M = \mathbb{0}_\mathrm{N}$

General idea of algorithm

1. Generic invariant form: $X = x_1\mathcal{A}_1 + \cdots + x_\mathrm{R}\mathcal{A}_\mathrm{R}$
2. $\mathcal{B}_m$ is solution of equation $X^2 - X = \mathbb{0}_\mathrm{N}$
3. $X^2 - X \iff$ set of quadratic polynomials:
   $$E(x_1, \ldots, x_\mathrm{R}) = \{E_1(x_1, \ldots, x_\mathrm{R}), \ldots, E_\mathrm{R}(x_1, \ldots, x_\mathrm{R})\}$$
4. $\mathcal{B}_m = b_{m,1}\mathcal{A}_1 + b_{m,2}\mathcal{A}_2 + \cdots + b_{m,\mathrm{R}}\mathcal{A}_\mathrm{R}$
   $[b_{m,1}, \ldots, b_{m,\mathrm{R}}]$ is solution of the system $E(x_1, \ldots, x_\mathrm{R}) = 0$
5. $\mathcal{A}_1 = \mathbb{1}_\mathrm{N} \Longrightarrow \mathrm{tr}\,\mathcal{B}_m = b_{m,1}\mathrm{N} \xRightarrow{\mathrm{tr}\,\mathcal{B}_m = d_m} b_{m,1} = d_m/\mathrm{N}$
6. $d_m \in [1, 2, \ldots, \mathrm{N} - 1]$

General outline

- inspect $d \in [1, 2, \ldots, \mathrm{N} - 1]$ in ascending order
- extract solutions $\{B_1, \ldots, B_k\}$ of $E(d/\mathrm{N}, x_2, \ldots, x_\mathrm{R}) = 0$
- add orthogonality conditions to polynomial system:
  $$E(x_1, \ldots, x_\mathrm{R}) \leftarrow E(x_1, \ldots, x_\mathrm{R}) \cup \{B_1X\} \cup \cdots \cup \{B_kX\}$$

Part I. Implemented in **C**

Input: $S = \{s_1, \ldots, s_K\}$ // generating set of permutations
Output: $E(x_1, \ldots, x_{\mathrm{R}})$ // idempotency polynomials
$O(b_1, \ldots, b_{\mathrm{R}}; x_1, \ldots, x_{\mathrm{R}})$ // orthogonality polynomials
`SplitRepresentation` // code in **Maple**

1: compute basis of centralizer ring $\mathcal{A}_1, \ldots, \mathcal{A}_{\mathrm{R}}$
2: compute multiplication table $\mathcal{A}_p\mathcal{A}_q = \sum_{r=1}^{\mathrm{R}} C_{pq}^r \mathcal{A}_r$
3: construct idempotency polynomials $E(x_1, \ldots, x_{\mathrm{R}})$
4: construct orthogonality polynomials $O(b_1, \ldots, b_{\mathrm{R}}; x_1, \ldots, x_{\mathrm{R}})$
5: construct code `SplitRepresentation` for processing polynomial data
6: **return** `SplitRepresentation` $(E(x_1, \ldots, x_{\mathrm{R}}),$
$O(b_1, \ldots, b_{\mathrm{R}}; x_1, \ldots, x_{\mathrm{R}}))$

Algorithm 1: `PreparePolynomialData`

Part II. Implemented in **Maple**

Input: $E(x_1, \ldots, x_\mathrm{R})$, $O(b_1, \ldots, b_\mathrm{R}; x_1, \ldots, x_\mathrm{R})$
Output: $\mathit{IrreducibleProjectors} = \left[(1, \mathcal{B}_1), \ldots, (d_m, \mathcal{B}_m) \ldots, (d_M, \mathcal{B}_M)\right]$

$\mathit{IrreducibleProjectors} \leftarrow \left[\left(1, \frac{1}{\mathrm{N}}\,[1, \ldots, 1]\right)\right]$ // trivial subrepresentation
$E(x_1, \ldots, x_\mathrm{R}) \leftarrow E(x_1, \ldots, x_\mathrm{R}) \cup O(1, \ldots, 1; x_1, \ldots, x_\mathrm{R})$
$\mathit{Sdim} \leftarrow 1$ // sum of dimensions, global variable
$D \leftarrow 0$ // current dimension, global variable
while $\mathit{Sdim} < \mathrm{N}$ **do**
    $D \leftarrow$ `NextRelevantDimension`(D)
    $\mathit{all\_solutions} \leftarrow$ `SolveAlgebraicSystem`$(E(D/\mathrm{N}, x_2, \ldots, x_\mathrm{R}))$
    if $\mathit{all\_solutions} \neq \varnothing$ **then**
        $h \leftarrow$ `NumberOfFreeParameters`$(\mathit{all\_solutions})$
        if $h = 0$ **then**
            for $\mathit{solution} \in \mathit{all\_solutions}$ **do**
                `UseSingleSolution`$(\mathit{solution})$
        else
            repeat
                $\mathit{solution} \leftarrow$ `PickBestSolution`$(\mathit{all\_solutions})$
                `UseSingleSolution`$(\mathit{solution})$
                $\mathit{all\_solutions} \leftarrow$ `SolveAlgebraicSystem`$(E(D/\mathrm{N}, x_2, \ldots, x_\mathrm{R}))$
            until $\mathit{all\_solutions} = \varnothing$
return $\mathit{IrreducibleProjectors}$

Algorithm 2: `SplitRepresentation`

Comments on `SplitRepresentation`

1. $D \leftarrow$ `NextRelevantDimension`(D)
   1. $D \leftarrow D + 1$ — simplest version
   2. **repeat** $D \leftarrow D + 1$ **until** $D \mid \mathrm{Ord}(\mathrm{G})$ — about 25% faster
   3. use character decomposition if known — most effective version
2. `UseSingleSolution`

   Input: $solution = [\beta_1, \ldots, \beta_{\mathrm{R}}]$

   1: $E(x_1, \ldots, x_{\mathrm{R}}) \leftarrow E(x_1, \ldots, x_{\mathrm{R}}) \cup O(\beta_1, \ldots, \beta_{\mathrm{R}}; x_1, \ldots, x_{\mathrm{R}})$

   2: $IrreducibleProjectors \leftarrow [IrreducibleProjectors, (D, solution)]$

   3: $Sdim \leftarrow Sdim + D$
3. number of free parameters $h > 0 \implies$ irreducible component of multiplicity $k : h = \lfloor k^2/2 \rfloor$

   General solution is a manifold of dimension h

   k mutually orthogonal solutions are constructed by procedure `PickBestSolution`

Our **C** + **Maple** vs **Magma**'s *MeatAxe*

3906-dimensional representation of exceptional group of Lie type $G_2(5)$

Splitting over $\mathrm{GF}(2)$ is demonstration of *MeatAxe* abilities in **Magma** documentation

`PreparePolynomialData` + `SplitRepresentation`:
Rank: 4. Suborbit lengths: $1, 30, 750, 3125$.

$$\underline{\mathbf{3906}} \cong \mathbf{1} \oplus \mathbf{930} \oplus \mathbf{1085} \oplus \mathbf{1890}$$

$$\mathcal{B}_{\mathbf{1}} = \frac{1}{3906}\sum_{k=1}^{4}\mathcal{A}_k$$

$$\mathcal{B}_{\mathbf{930}} = \frac{5}{21}\left(\mathcal{A}_1 + \frac{3}{10}\mathcal{A}_2 + \frac{1}{50}\mathcal{A}_3 - \frac{1}{125}\mathcal{A}_4\right)$$

$$\mathcal{B}_{\mathbf{1085}} = \frac{5}{18}\left(\mathcal{A}_1 - \frac{1}{5}\mathcal{A}_2 + \frac{1}{25}\mathcal{A}_3 - \frac{1}{125}\mathcal{A}_4\right)$$

$$\mathcal{B}_{\mathbf{1890}} = \frac{15}{31}\left(\mathcal{A}_1 - \frac{1}{30}\mathcal{A}_2 - \frac{1}{30}\mathcal{A}_3 + \frac{1}{125}\mathcal{A}_4\right)$$

Time **C**: 1.14 sec. Time **Maple**: 0.8 sec.

Magma fails to split the representation over $\mathbb{Q}$

$\mathrm{Ord}(G_2(5)) = 5859000000 = 2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31 \Longrightarrow$ let's try $\mathrm{GF}(11)$

```
> load "g25";
Loading "/opt/magma.21-1/libs/pergps/g25"
The Lie group G( 2, 5 ) represented as a permutation
group of degree 3906.
Order: 5 859 000 000 = 2^6 * 3^3 * 5^6 * 7 * 31.
Group: G
> time Constituents(PermutationModule(G,GF(11)));
[
    GModule of dimension 1 over GF(11),
    GModule of dimension 930 over GF(11),
    GModule of dimension 1085 over GF(11),
    GModule of dimension 1890 over GF(11)
]
Time: 282.060
```

Important group constructs

- Schur multiplier: $\mathrm{M}(\mathsf{G}) = H_2(\mathsf{G}, \mathbb{Z})$

  for $\mathsf{C}_n \cong \mathrm{M}(\mathsf{G})/N$

  $n.\mathsf{G}$ denotes a central extension or covering group of G:
  $1 \to \mathsf{C}_n \to n.\mathsf{G} \to \mathsf{G} \to 1$

- Outer automorphism group: $\mathrm{Out}(\mathsf{G}) = \mathrm{Aut}(\mathsf{G})/\mathrm{Inn}(\mathsf{G})$

  for $H \cong \mathrm{Out}(\mathsf{G})/N \longrightarrow \mathsf{G} \rtimes H$

  for $H = \mathsf{C}_n$ the notation $\mathsf{G}:n$ is used

Examples of computations

Timings: PC with 3.30GHz Intel Core i3 2120 CPU

Generators of group representations:
section "Sporadic groups" ATLAS of Finite Group Representations
http://brauer.maths.qmul.ac.uk/Atlas/v3/

2016**-dimensional representation of Mathieu group cover** $3.M_{22}$
Rank: 16 Suborbit lengths: $1^3, 55^3, 66^3, 165^4, 330^3$

$$\underline{\mathbf{2016}} \cong \mathbf{1} \oplus \mathbf{21}_\alpha \oplus \mathbf{21}_\beta \oplus \overline{\mathbf{21}_\beta} \oplus \mathbf{55} \oplus \mathbf{105}_+ \oplus \overline{\mathbf{105}_+} \oplus \mathbf{105}_- \oplus \overline{\mathbf{105}_-}$$
$$\oplus \mathbf{154} \oplus \mathbf{210}_\alpha \oplus \mathbf{210}_\beta \oplus \overline{\mathbf{210}_\beta} \oplus \mathbf{231}_\alpha \oplus \mathbf{231}_\beta \oplus \overline{\mathbf{231}_\beta}$$

$$\mathcal{B}_{\mathbf{1}} = \frac{1}{2016} \sum_{k=1}^{16} \mathcal{A}_k$$

$$\mathcal{B}_{\mathbf{21}_\alpha} = \frac{1}{96}\left(\mathcal{A}_1 - \frac{5}{11}\mathcal{A}_2 + \frac{3}{11}\mathcal{A}_3 - \frac{1}{11}\mathcal{A}_4 + \frac{3}{11}\mathcal{A}_5 - \frac{1}{11}\mathcal{A}_6 - \frac{1}{11}\mathcal{A}_7 - \frac{1}{11}\mathcal{A}_8 - \frac{1}{11}\mathcal{A}_9 - \frac{1}{11}\mathcal{A}_{10} + \frac{3}{11}\mathcal{A}_{11} + \frac{3}{11}\mathcal{A}_{12} - \frac{5}{11}\mathcal{A}_{13} - \frac{5}{11}\mathcal{A}_{14} + \mathcal{A}_{15} + \mathcal{A}_{16}\right)$$

$$\mathcal{B}_{\mathbf{21}_\beta} = \frac{1}{96}\left\{\mathcal{A}_1 + \frac{4}{11}\mathcal{A}_2 + \frac{3}{11}\mathcal{A}_3 - \frac{1}{11}\mathcal{A}_4 - \frac{4}{11}\mathcal{A}_6 + \frac{2}{11}\left(1+\mathbf{i}\sqrt{3}\right)\mathcal{A}_7 + \frac{2}{11}\left(1-\mathbf{i}\sqrt{3}\right)\mathcal{A}_8 + \frac{1}{22}\left(1+\mathbf{i}\sqrt{3}\right)\mathcal{A}_9 + \frac{1}{22}\left(1-\mathbf{i}\sqrt{3}\right)\mathcal{A}_{10} - \frac{3}{22}\left(1+\mathbf{i}\sqrt{3}\right)\mathcal{A}_{11} - \frac{3}{22}\left(1-\mathbf{i}\sqrt{3}\right)\mathcal{A}_{12} - \frac{2}{11}\left(1+\mathbf{i}\sqrt{3}\right)\mathcal{A}_{13} - \frac{2}{11}\left(1-\mathbf{i}\sqrt{3}\right)\mathcal{A}_{14} - \frac{1}{2}\left(1+\mathbf{i}\sqrt{3}\right)\mathcal{A}_{15} - \frac{1}{2}\left(1-\mathbf{i}\sqrt{3}\right)\mathcal{A}_{16}\right\}$$

$$\mathcal{B}_{\mathbf{55}} = \frac{55}{2016}\left(\mathcal{A}_1 + \frac{13}{55}\mathcal{A}_2 - \frac{1}{55}\mathcal{A}_3 - \frac{1}{55}\mathcal{A}_4 + \frac{13}{55}\mathcal{A}_5 - \frac{3}{11}\mathcal{A}_6 - \frac{3}{11}\mathcal{A}_7 - \frac{3}{11}\mathcal{A}_8 - \frac{1}{55}\mathcal{A}_9 - \frac{1}{55}\mathcal{A}_{10} - \frac{1}{55}\mathcal{A}_{11} - \frac{1}{55}\mathcal{A}_{12} + \frac{13}{55}\mathcal{A}_{13} + \frac{13}{55}\mathcal{A}_{14} + \mathcal{A}_{15} + \mathcal{A}_{16}\right)$$

$$\mathcal{B}_{\mathbf{105}_\pm} = \frac{5}{96}\left\{\mathcal{A}_1 - \frac{2}{55}\left(1 \pm \sqrt{33}\right)\mathcal{A}_2 + \frac{1}{55}\left(4 \pm \frac{\sqrt{33}}{3}\right)\mathcal{A}_3 + \frac{1}{110}\left(1 \mp \frac{\sqrt{33}}{3}\right)\mathcal{A}_4 + \frac{1}{22}\left(3 \mp \frac{\sqrt{33}}{3}\right)\mathcal{A}_6 - \frac{1}{44}\left(3 \mp \frac{\sqrt{33}}{3} - \mathbf{i}\left(\sqrt{11} \mp 3\sqrt{3}\right)\right)\mathcal{A}_7 - \frac{1}{44}\left(3 \mp \frac{\sqrt{33}}{3} + \mathbf{i}\left(\sqrt{11} \mp 3\sqrt{3}\right)\right)\mathcal{A}_8\right.$$

$$
\begin{aligned}
&- \frac{1}{220}\left(1 \mp \frac{\sqrt{33}}{3} - \mathbf{i}\left(\sqrt{11} \mp \sqrt{3}\right)\right)\mathcal{A}_{9}\\
&- \frac{1}{220}\left(1 \mp \frac{\sqrt{33}}{3} + \mathbf{i}\left(\sqrt{11} \mp \sqrt{3}\right)\right)\mathcal{A}_{10}\\
&- \frac{1}{110}\left(4 \pm \frac{\sqrt{33}}{3} + \mathbf{i}\left(\sqrt{11} \pm 4\sqrt{3}\right)\right)\mathcal{A}_{11}\\
&- \frac{1}{110}\left(4 \pm \frac{\sqrt{33}}{3} - \mathbf{i}\left(\sqrt{11} \pm 4\sqrt{3}\right)\right)\mathcal{A}_{12}\\
&+ \frac{1}{55}\left(1 \pm \sqrt{33} + \mathbf{i}\left(3\sqrt{11} \pm \sqrt{3}\right)\right)\mathcal{A}_{13}\\
&+ \frac{1}{55}\left(1 \pm \sqrt{33} - \mathbf{i}\left(3\sqrt{11} \pm \sqrt{3}\right)\right)\mathcal{A}_{14}\\
&-\frac{1}{2}\left(1 \pm \mathbf{i}\sqrt{3}\right)\mathcal{A}_{15} - \frac{1}{2}\left(1 \mp \mathbf{i}\sqrt{3}\right)\mathcal{A}_{16}\Big\}
\end{aligned}
$$

$$\mathcal{B}_{\mathbf{154}} = \frac{11}{144}\left(\mathcal{A}_1 + \frac{7}{55}\mathcal{A}_2 + \frac{3}{55}\mathcal{A}_3 - \frac{3}{55}\mathcal{A}_4 - \frac{1}{11}\mathcal{A}_5 + \frac{1}{11}\mathcal{A}_6 + \frac{1}{11}\mathcal{A}_7 + \frac{1}{11}\mathcal{A}_8 - \frac{3}{55}\mathcal{A}_9 - \frac{3}{55}\mathcal{A}_{10} + \frac{3}{55}\mathcal{A}_{11} + \frac{3}{55}\mathcal{A}_{12} + \frac{7}{55}\mathcal{A}_{13} + \frac{7}{55}\mathcal{A}_{14} + \mathcal{A}_{15} + \mathcal{A}_{16}\right)$$

$$\mathcal{B}_{\mathbf{210}_\alpha} = \frac{5}{48}\left(\mathcal{A}_1 - \frac{3}{55}\mathcal{A}_2 + \frac{1}{165}\mathcal{A}_3 + \frac{7}{165}\mathcal{A}_4 - \frac{7}{55}\mathcal{A}_5 - \frac{1}{11}\mathcal{A}_6 - \frac{1}{11}\mathcal{A}_7 - \frac{1}{11}\mathcal{A}_8 + \frac{7}{165}\mathcal{A}_9 + \frac{7}{165}\mathcal{A}_{10} + \frac{1}{165}\mathcal{A}_{11} + \frac{1}{165}\mathcal{A}_{12} - \frac{3}{55}\mathcal{A}_{13} - \frac{3}{55}\mathcal{A}_{14} + \mathcal{A}_{15} + \mathcal{A}_{16}\right)$$

$$
\begin{aligned}
\mathcal{B}_{\mathbf{210}_\beta} = \frac{5}{48}\Big\{ & \mathcal{A}_1 - \frac{1}{15}\mathcal{A}_3 - \frac{1}{15}\mathcal{A}_4 \\
& + \frac{1}{30}\left(1+\mathbf{i}\sqrt{3}\right)\mathcal{A}_9 + \frac{1}{30}\left(1-\mathbf{i}\sqrt{3}\right)\mathcal{A}_{10} \\
& + \frac{1}{30}\left(1+\mathbf{i}\sqrt{3}\right)\mathcal{A}_{11} + \frac{1}{30}\left(1-\mathbf{i}\sqrt{3}\right)\mathcal{A}_{12} \\
& - \frac{1}{2}\left(1+\mathbf{i}\sqrt{3}\right)\mathcal{A}_{15} - \frac{1}{2}\left(1-\mathbf{i}\sqrt{3}\right)\mathcal{A}_{16}\Big\} \\
\mathcal{B}_{\mathbf{231}_\alpha} = \frac{11}{96}\Big(& \mathcal{A}_1 - \frac{3}{55}\mathcal{A}_2 - \frac{1}{15}\mathcal{A}_3 + \frac{1}{165}\mathcal{A}_4 + \frac{1}{11}\mathcal{A}_5 + \frac{1}{11}\mathcal{A}_6 \\
& + \frac{1}{11}\mathcal{A}_7 + \frac{1}{11}\mathcal{A}_8 + \frac{1}{165}\mathcal{A}_9 + \frac{1}{165}\mathcal{A}_{10} - \frac{1}{15}\mathcal{A}_{11} \\
& - \frac{1}{15}\mathcal{A}_{12} - \frac{3}{55}\mathcal{A}_{13} - \frac{3}{55}\mathcal{A}_{14} + \mathcal{A}_{15} + \mathcal{A}_{16}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_{\mathbf{231}_\beta} = \frac{11}{96}\Big\{ & \mathcal{A}_1 - \frac{1}{33}\mathcal{A}_3 + \frac{2}{33}\mathcal{A}_4 - \frac{1}{11}\mathcal{A}_6 \\
& + \frac{1}{22}\left(1 + \mathbf{i}\sqrt{3}\right)\mathcal{A}_7 + \frac{1}{22}\left(1 - \mathbf{i}\sqrt{3}\right)\mathcal{A}_8 \\
& - \frac{1}{33}\left(1 + \mathbf{i}\sqrt{3}\right)\mathcal{A}_9 - \frac{1}{33}\left(1 - \mathbf{i}\sqrt{3}\right)\mathcal{A}_{10} \\
& + \frac{1}{66}\left(1 + \mathbf{i}\sqrt{3}\right)\mathcal{A}_{11} + \frac{1}{66}\left(1 - \mathbf{i}\sqrt{3}\right)\mathcal{A}_{12} \\
& - \frac{1}{2}\left(1 + \mathbf{i}\sqrt{3}\right)\mathcal{A}_{15} - \frac{1}{2}\left(1 - \mathbf{i}\sqrt{3}\right)\mathcal{A}_{16}\Big\}
\end{aligned}
$$

Time **C**: 3 sec. Time **Maple**: 1 h 10 min 48 sec.

Several decompositions in concise form

- 1980-dimensional representation of Mathieu group cover $6.M_{22}$
  Rank: 17. Suborbit lengths: $1^6, 14^3, 84^3, 336^5$.

$$\underline{\mathbf{1980}} \cong \mathbf{1} \oplus \mathbf{21}_\alpha \oplus \mathbf{21}_\beta \oplus \overline{\mathbf{21}_\beta} \oplus \mathbf{55} \oplus \mathbf{99}_\alpha \oplus \mathbf{99}_\beta \oplus \overline{\mathbf{99}_\beta} \oplus \mathbf{105}_+ \oplus \overline{\mathbf{105}_+} \oplus \mathbf{105}_- \oplus \overline{\mathbf{105}_-} \oplus \mathbf{120} \oplus \mathbf{154} \oplus \mathbf{210} \oplus \mathbf{330} \oplus \overline{\mathbf{330}}$$

  Time **C**: 2 sec. Time **Maple**: 8 h 41 min 1 sec.

- 29155-dimensional representation of Held group He
  Rank: 12. Suborbit lengths:
  $1, 90, 120, 384, 960^2, 1440, 2160, 2880^2, 5760, 11520$.

$$\underline{\mathbf{29155}} \cong \mathbf{1} \oplus \mathbf{51} \oplus \overline{\mathbf{51}} \oplus \mathbf{680} \oplus (\mathbf{1275} \oplus \mathbf{1275}) \oplus \mathbf{1920} \oplus \mathbf{4352} \oplus \mathbf{7650} \oplus \mathbf{11900}$$

  Time **C**: 5 min 41 sec. Time **Maple**: 15 sec.

- 66825-dimensional representation of McLaughlin group cover $3.McL$
  Rank: 14. Suborbit lengths: $1^3, 630, 2240^3, 5040^3, 8064^3, 20160$.

  $$\underline{\mathbf{66825}} \cong \mathbf{1} \oplus \mathbf{252} \oplus \mathbf{1750} \oplus \mathbf{2772} \oplus \overline{\mathbf{2772}} \oplus \mathbf{5103}_\alpha \oplus \mathbf{5103}_\beta \oplus \overline{\mathbf{5103}_\beta}$$
  $$\oplus \mathbf{5544} \oplus \mathbf{6336} \oplus \overline{\mathbf{6336}} \oplus \mathbf{8064} \oplus \overline{\mathbf{8064}} \oplus \mathbf{9625}$$

  Time **C**: 39 min 36 sec. Time **Maple**: 14 min 11 sec.

- 98280-dimensional representation of Suzuki group cover $3.Suz$
  Rank: 14. Suborbit lengths: $1^3, 891^3, 2816^3, 5940, 19008, 20736^3$.

  $$\underline{\mathbf{98280}} \cong \mathbf{1} \oplus \mathbf{78} \oplus \overline{\mathbf{78}} \oplus \mathbf{143} \oplus \mathbf{364} \oplus \mathbf{1365} \oplus \overline{\mathbf{1365}} \oplus \mathbf{4290} \oplus \overline{\mathbf{4290}}$$
  $$\oplus \mathbf{5940} \oplus \mathbf{12012} \oplus \mathbf{14300} \oplus \mathbf{27027} \oplus \overline{\mathbf{27027}}$$

  Time **C**: 2 h 36 min 29 sec. Time **Maple**: 7 min 41 sec.