Bounds on Betti Numbers of Tropical Prevarieties

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Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus$, $\otimes$.

If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.

If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.

Examples • $\mathbb{Z}^+ := \{0 \leq a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;

• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;

• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \leq j \leq n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.

Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j\{Q_j\}$;

$x = (x_1, \ldots, x_n)$ is a **tropical zero** of f if minimum $\min_j\{Q_j\}$ is attained for at least two different values of j.

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Historical sources of the tropical algebra

Logarithmic scaling of the reals (mathematical physics)

Define two operations on positive reals, replacing addition and multiplication:

$a, b \to t \cdot \log(\exp(a/t) + \exp(b/t)), \quad \lim_{t \to 0} = \max\{a, b\}$

$a, b \to t \cdot \log(\exp(a/t) \cdot \exp(b/t)) = a + b$

Thus, the "dequantization" of the logarithmic scaling is a tropical semi-ring

Solving systems of polynomial equations in Puiseux series (algebraic geometry)

The field of Puiseux series $F((t^{1/\infty})) \ni a_0 \cdot t^{i/q} + a_1 \cdot t^{(i+1)/q} + \cdots, \ 0 < q \in \mathbb{Z}$ over an algebraically closed field F is algebraically closed. In the (Newton) algorithm for solving a system of polynomial equations $f_i(X_1, \ldots, X_n) = 0, \ 1 \le i \le k$ with $f_i \in F((t^{1/\infty}))[X_1, \ldots, X_n]$ in Puiseux series the leading exponents i_j/q_j in $X_j = a_{0j} \cdot t^{i_j/q_j} + \cdots$ satisfy a tropical polynomial system (due to cancelation of the leading terms).

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Minimal weights of paths in a graph (computer science)

For a graph with weights w_{ij} on edges (i, j) for any k to compute for each pair of vertices i, j the minimal weight of paths between i and j. This is equivalent to computing the tropical k-th power of matrix (w_{ij}).

Scheduling

Let several jobs i should be executed by means of several machines j with times of execution t_{ij}. The restrictions like that job i_0 should be executed after job i are imposed. Denoting by unknown x_{ij} a starting moment of execution of i by j, the latter restriction is expressed as $x_{i_0,j_0} \geq \min_j\{x_{ij} + t_{ij}\}$. Another sort of restrictions is that a machine can't execute two jobs simultaneously, i. e. $x_{i_1,j} \geq x_{ij} + t_{ij}$. It leads to a system of min-plus linear inequalities, the problem being equivalent to tropical linear systems.

This approach is employed in scheduling of Dutch and Korean railways.

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Tropical Varieties and Prevarieties

$K = \mathbb{C}((t^{1/\infty})) = \{c = c_0 t^{i_0/q} + c_1 t^{(i_0+1)/q} + \cdots\}$

is a field of Puiseux series where $i_0 \in \mathbb{Z}$, $1 \leq q \in \mathbb{Z}$.

Consider an ideal $I \subset K[X_1, \ldots, X_n]$, the variety of its solutions $U(I) \subset K^n$.

Tropicalization $Trop(c) = i_0/q$, $Trop(0) = \infty$.

The closure in the Euclidean topology $V := \overline{Trop(U(I))} \subset \mathbb{R}^n$ is called the **tropical variety** of I.

$\overline{Trop(U(f))} \subset \mathbb{R}^n$ is a **tropical hypersurface** where $f \in K[X_1, \ldots, X_n]$.

$V(f_1, \ldots, f_k) := \overline{Trop(U(f_1))} \cap \cdots \cap \overline{Trop(U(f_k))}$ is a **tropical prevariety**. Any tropical variety is a tropical prevariety, but not necessary vice versa.

Any tropical prevariety is a polyhedral complex. Moreover, when ideal I is prime the tropical variety $\overline{Trop(U(I))}$ has at any point the same local dimension equal $\dim I$.

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Denote by $P_l \subset \mathbb{R}^n$ Newton polytope of f_i, $1 \le i \le k$.

Theorem

The number of faces of all dimensions of a tropical prevariety $V = V(f_1, \dots, f_k)$ does not exceed $(2^{n+1} - 1) \cdot n! \cdot Vol_n(P_1 + \cdots + P_k)$.

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*(**Weak inequality of discrete Morse theory, R. Forman**). l-th Betti number (the rank of l-th homology group) of V is less or equal to the number of l-dimensional faces of V.*

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For $\mathrm{trdeg}(f_i) \leq d,\ 1 \leq i \leq k$ *the sum of Betti numbers of* V *is less than* $(2^{n+1} - 1) \cdot (kd)^n$.

Compare with classical polynomials $h_1, \ldots, h_k \in \mathbb{R}[X_1, \ldots, X_n]$ defining a semi-algebraic set $W := \{x \in \mathbb{R}^n : h_i(x) \geq 0,\ 1 \leq i \leq k\}$.

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Proof of the bound on the number of faces of a tropical prevariety: dual polyhedron

Denote $f_i := \bigoplus_J a_{J,i} \otimes X^{\otimes J}$, $a_{J,i} \in \mathbb{R}$, $J = (j_1, \dots, j_n)$, $1 \le i \le k$. Extended Newton polytope $Q_i \subset \mathbb{R}^{n+1}$ of f_i is the convex hull of points $(J, a_{J,i})$. Denote by Q the bottom (i. e. the lowest with respect to the last coordinate points) of $Q_1 + \cdots + Q_k$ together with all the rays emanating upwards from the bottom. Denote the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ along the last coordinate. Then $\pi(Q) = \pi(Q_1 + \cdots + Q_k) = P_1 + \cdots + P_k$.

For a face F of Q without vertical rays its dual $G(F)$ is defined as the set of all supporting hyperplanes H without vertical lines to Q such that $H \cap Q = F$. Then $G(F)$ is identified with a face of the dual polyhedron to Q, and $\dim F + \dim G(F) = n$. Observe that F is representable as a Minkowski sum $F = F_1 + \cdots + F_k$ where F_i is a face of (the bottom) of Q_i such that any $H \in G(F)$ is a supporting hyperplane for Q_i and $H \cap Q_i = F_i$. We say that a face F (without vertical rays) is *tropical* if $\dim F_i \ge 1$, $1 \le i \le k$. Then $V(f_1, \dots, f_k)$ coincides with the union of polyhedra $G(F)$ for all tropical faces F.

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Proof of the bound on the number of faces of a tropical prevariety: triangulation and volume estimating

Decompose each n-dimensional face of Q without vertical rays into n-dimensional closed simplices without adding new vertices. The number of all subsimplices of these simplices is not less than the total number of faces in Q without vertical rays, which in its turn, is not less than the total number of faces in $V(f_1, \dots, f_k)$.

Since for each n-dimensional simplex S in the decomposition its projection $\pi(S) \subset \mathbb{R}^n$ has integer vertices, we get $Vol_n(\pi(S)) \geq 1/n!$. Therefore, the number of all n-dimensional simplices in the decomposition does not exceed $n! \cdot Vol_n(P_1 + \cdots + P_k)$. To complete the proof it remains to notice that the number of all subsimplices of an n-dimensional simplex equals $2^{n+1} - 1$.

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Explicit representation of a tropical prevariety as a polyhedral complex

Assume that each tropical polynomial $f_i = \min\{L_{i,1}, \dots, L_{i,m}\}$, $1 \le i \le k$ is m-sparse, so has at most m monomials, where $L_{i,j}$ are linear polynomials. For any subset $B \subset D := \{(i,j) : 1 \le i \le k,\ 1 \le j \le m\}$ consider the polyhedron U_B consisting of points $x \in \mathbb{R}^n$ such that for each $1 \le i \le k$

$$\min_{1 \le j \le m} \{L_{i,j}(x)\} = L_{i,j_0}(x),\ (i, j_0) \in B,$$

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Tropical prevariety $V(f_1, \dots, f_k)$ is the union of all U_B such that for each $1 \le i \le k$ there exist $1 \le j_2 < j_3 \le m$ with $(i, j_2), (i, j_3) \in B$.

Moreover, U_B constitute a polyhedral complex: the faces of every U_B are also some U_{B_l}, and each intersection of the closures $\overline{U_{B_l}} \cap \overline{U_{B_s}}$ equals $\overline{U_{B_q}}$ for suitable q.

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Betti numbers for sparse tropical polynomials

Consider arrangement A of at most $k \cdot \binom{m}{2}$ hyperplanes of the form $L_{i,j_1} = L_{i,j_2}$, $1 \leq i \leq k$, $1 \leq j_1 < j_2 \leq m$. For any $B \subset D$ polyhedron U_B is a face of A. Therefore, by the Weak Morse Inequality we get

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The sum of Betti numbers of a tropical prevariety $V(f_1, \dots, f_k)$ defined by m-sparse tropical polynomials $f_1, \dots, f_k$ does not exceed

$$n \cdot 2^n \cdot \binom{k \cdot \binom{m}{2}}{n}$$

owing to the bound on the number of faces of an arrangement due to **T. Zaslavski**.

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