Reverse Decomposition of Unipotents

Nikolai Vavilov

Abstract. Decomposition of unipotents gives short polynomial expressions of the conjugates of elementary genetators as products of elementaries. It turns out that with some minor twist the decomposition of unipotents can be read backwards, to give very short polynomial expressions of elementary generators in terms of elementary conjugates of an arbitrary matrix and its inverse. For absolute elementary subgroups of classical groups this was recently observed by Raimund Preusser. I discuss various generalisations of these results for exceptional groups, at the relative level, and possible applications.

Decomposition of unipotents [6] was first proposed by Alexei Stepanov for $GL(n,R)$ in 1987, immediately generalised to other split classical groups by the present author, and then further developed in other contexts by a number of authors, see [8, 5, 2] for many further references.

In its simplest form, it can be viewed as a constructive version of the normality of the elementary subgroup. Namely, let Φ be a root system, R be an arbitrary commutative ring with 1, and $G(\Phi,R)$ be the simply connected Chevalley group of type Φ over R. Further, fix a split maximal torus $T(\Phi,R)$ of $G(\Phi,R)$ and the corresponding elementary generators $x_\alpha(\xi)$, where $\alpha\in\Phi$, $\xi\in R$. Let $E(\Phi,R)$ be the elementary subgroup spanned by all these elementary generators.

Then decomposition of unipotents provides explicit polynomial formulae expressing the conjugate $gx_\alpha(\xi)g^{-1}$ of an elementary generator by an arbitrary matrix $g\in G(\Phi,R)$ as a product of elementaries. Thus, for instance, for the groups of types E_6 and E_7 any such conjugate is the product of at most $4\cdot 27\cdot 16$ and $4\cdot 56\cdot 27$ elementary generators, respectively [7]

Another central classical result in the structure theory of Chevalley groups is description of their normal subgroups, or rather their subgroups normalised by the elementary group $E(\Phi,R)$. What would be an explicit contructive version of that? Until very recently, this was only known in some very special cases. Thus, for $SL(n,\mathbb{Z})$, $n\geq 3$, Joel Brenner [1] established that for an arbitrary matrix $g\in SL(n,\mathbb{Z})$ an elementary transvection $t_{ij}(\xi)$, where ξ belongs to the level of g, is a bounded product of conjugates of g and g^{-1}. Brenner's proof used the theory of elementary divisors, and even generalisations to other groups over PID were not immediate at all. And of course, there was no hope whatsoever to write such similar formulae for arbitrary commutative rings.

Thus, we were seriously perplexed, when we've first seen the preprints of [3, 4] in Summer 2017. The calculations in [3] start in exactly the same way as in [6], so predictably our assessment of these papers came through the following three stages: 1) There must be nothing new as compared with [6], 2) Gosh, why is it true at all? 3) It is a fantastic breakthrough in the structure theory of algebraic-like groups!

Technically, the twist introduced by Raimund Preusser in the decomposition of unipotents seems to be minor. It consists in expressing a conjugate of an elementary generator not as a product of factors sitting in proper parabolics of certain types, but rather sitting in the products of these parabolics by something small in the unipotent radicals of the opposite parabolics. We were aware of the idea itself [5], but have never appreciated the whole significance of this apparently small variation.

In fact, it allows to reduce degree of the resulting polynomials, and thus both to completely avoid the cumbersome "main lemma", establishing that the coefficients of the occuring polynomials generate the unit ideal, and drastically lower the depth of commutators. In particular, Preusser's idea allows to prove analogues of Brenner's lemma for groups of all types over arbitrary commutative rings, and more.

Immediately after understanding this idea, we were able to generalise it to exceptional groups as well, and to other situations. In particular, it can be derived that for an arbitrary commutative ring R and an arbitrary matrix $g \in G(\Phi, R)$ one can write explicit formulae, expressing an elementary generator $x_\alpha(\xi)$, where ξ belongs to the level of g, as products of at most $8 \cdot dim(G)$ elementary conjugates of g and g^{-1}.

I discuss further development of this idea, such as our joint paper with Zhang Zuhong, where we write similar formulae at the relative level, expressing elementary generators as products of conjugates of g and g^{-1} by elements of the relative elementary subgroups $E(\Phi, R, I)$, corresponding to an ideal $I \trianglelefteq R$. This result has significant applications to the description of subnormal subgroups of $G(\Phi, R)$, etc.

I sketch further imminent applications of these ideas, to description of various classes of intermediate subgroups, the values of word maps, etc.

References

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Nikolai Vavilov
Department of mathematics and Mechanics
St. Petersburg State University
St. Petersburg, Russia
e-mail: `nikolai-vavilov@yandex.ru`