Links from second-order Fuchsian equations to first-order linear systems.

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From the equation to the system.

Consider a scalar linear equation of the second order:

$$\psi'' + P\psi' + Q\psi = 0, \quad P, Q \text{ are rational.} \tag{1}$$

We can rewrite it as a first-order system:

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}.$$

Classical results tell us that if the scalar equation (1) is Fichsian, there is such a polynomial gauge transformation $(\psi, \psi')^T \to g(z)(\psi, \psi')^T =: \vec{\psi}$ that the resulting system

$$\vec{\psi}' = A(z)\vec{\psi}, \quad A = g \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} g^{-1} + g'g^{-1}$$

becomes Fuchsian, i.e. $A(z)dz$ has only simple poles on $\overline{\mathbb{C}}$.

From the system to the equation.

If we start from the system $\vec{\psi}' = A(z)\vec{\psi}$, we can exclude the second component of $\vec{\psi} = (\psi, \psi_2)^T$ using $\vec{\psi}'' = (A' + A^2)\vec{\psi}$. The result is the scalar equation $\psi'' + P\psi' + Q\psi = 0$, where

$$P = -\log' A_{12} - \mathrm{tr} A, \quad Q = \det A - A'_{11} + A_{11} \log' A_{12}.$$

How to restore the Fuchsian system if the scalar equation is a Heun equation (Fuchsian eqn. with four singular points)?

Characteristic exponents.

The roots $\rho_{1,2} = \rho_{1,2}(z_k)$ of the quadratic equation

$$\rho(\rho-1)+\rho\,\mathrm{Res}\,|_{z=z_k}P+\mathrm{Res}\,|_{z=z_k}\,((z-z_k)Q) = 0, k = 1, 2, 3.$$

are called characteristic exponents at z_k. They are connected with the eigenvalues Θ'_k, Θ_k of the residues $A^{(k)}$ of $A(z) = \sum_{k=1}^{3} \frac{A^{(k)}}{z-z_k}$:

$$\{\rho_1, \rho_2\} = \{\Theta'_k, \Theta_k + s_k + 1\},$$

where s_k is an order of $A_{12}(z)$ at z_k: $A_{12}(z) \sim (z-z_k)^{s_k}$.

s-homothopic transformations.

The transformation $\psi \to \prod_{k=1}^{3}(z-z_k)^{\alpha_k}\psi, \sum_k \alpha_k = -\alpha_\infty$ shifts the exponents and eigenvalues:
$\{\Theta'_k, \Theta_k\} \to \{\Theta'_k + \alpha_k, \Theta_k + \alpha_k\}$,
so we can put $\Theta_k = 0,\ k = 1, 2, 3$. In this case the formulas become simpler, and the matrices-residues $A^{(k)}$ for the system corresponding Heun equation

$$\sigma(z)\frac{d^2}{dz^2}\psi + \tau(z)\frac{d}{dz}\psi + (\alpha\beta z - \lambda)\psi = 0. \qquad (2)$$

$$\sigma(z) = \prod_{j=1}^{3}(z-z_j), \quad \tau(z) = \sum_{j=1}^{3}(1-\Theta_j)\sigma_j(z), \quad \sigma_j = \sigma(z)/(z-z_j),$$

can be calculated straightforwardly:

Degenerated residues: $\det A^{(k)} = 0$, k=1,2,3.

$$A^{(1)} = \begin{pmatrix} 0 & 0 \\ h & \Theta_1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} \frac{\alpha}{1-t} & -t \\ \frac{\alpha}{t(1-t)}\left(\frac{\alpha}{1-t} - \Theta_2\right) & \Theta_2 - \frac{\alpha}{1-t} \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} \frac{-t\alpha}{1-t} & t \\ \frac{-\alpha}{1-t}\left(\frac{t\alpha}{1-t} + \Theta_3\right) & \Theta_3 + \frac{t\alpha}{1-t} \end{pmatrix}, A^{(\infty)} = -\sum_k A^{(k)}$$

$$h := \lambda / t(t-1)$$

Quotion with respect to $\mathrm{GL}(2, \mathbb{C})$.

The equivalent sets $\{A^{(k)}\} \sim \{g^{-1}A^{(k)}g\}$, $g \in \mathrm{GL}(2)$ have unique representative such that

$$A^{(1)} = \begin{pmatrix} \Theta_1' & \star \\ 0 & \Theta_1 \end{pmatrix}, A^{(2)} = \begin{pmatrix} \star & -1 \\ \star & \star \end{pmatrix}, A^{(3)} = \begin{pmatrix} \Theta_3' & 0 \\ \star & \Theta_3 \end{pmatrix}.$$

If the eigenvalues are given, all the matrix elements of these matrices are defined by $A^{(\infty)} = -\sum_k A^{(k)}$, and there are no restrictions on $A^{(\infty)}$.

All other scalar equations from the class form one-parametric set sweeped out by the action of

$$g_p = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad \{A^{(k)}\} \to \{g_p^{-1}A^{(k)}g_p\}.$$

The enumeration of Fuchsian singularities leads to the enumeration of $A^{(k)}$.

Discrimination of the equations without apparent singulaties.

The formulae

$$P = -\log' A_{12} - \mathrm{tr} A, \quad Q = \det A - A'_{11} + A_{11} \log' A_{12}$$

show that P, Q have singularities at the zeros of A_{12}. These are the apparent singularities. Heun equation has just four singularities at z_k, and has no apparent singularities. It implies the coinciding both zeros of $A_{12} dz$ with some z_k.

Two residues, say $A^{(3)}, A^{(\infty)}$ are lower-triangular. It gives

The normalized set of $A^{(k)}$ for Heun equation.

$$A^{(1)} = \begin{pmatrix} \Theta_1' & 1 \\ 0 & \Theta_1 \end{pmatrix}, A^{(2)} = \begin{pmatrix} \Theta_2' + \Theta_\Sigma & -1 \\ \Theta_\Sigma(\Theta_\Sigma + \Theta_2'') & \Theta_2 - \Theta_\Sigma \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} \Theta_3' & 0 \\ h & \Theta_3 \end{pmatrix}, A^{(\infty)} = \begin{pmatrix} \Theta_4' & 0 \\ -(\Theta_\Sigma(\Theta_\Sigma + \Theta_3'') + h) & \Theta_4 \end{pmatrix},$$

$\Theta_j'' = \Theta_j' - \Theta_j$, $\Theta_\Sigma := \sum_{j=1}^4 \Theta_j$, $h \in \mathbb{C}$ is unique parameter, it is the accessory parameter. This normalization is more suitable compared with the conventional condition in isomonodromic property demanding diagonal A^∞.

Heun equation for the case $\operatorname{tr} A^{(k)} = 0$.

Formulae for the coefficients become simpler if we set $\Theta + \Theta' = 0$, $z_1 = 0, z_2 = 1, z_3 = t, z_4 = \infty$, namely

$$\psi'' + \left(\frac{1}{z} + \frac{1}{z-1}\right)\psi' + \left(\frac{h}{z(z-1)(z-t)} + \widetilde{Q}\right)\psi = 0,$$

where

$$\widetilde{Q} = -\frac{\Theta_3(2(\Theta_2 - \Theta_\Sigma) - 1)}{(z-1)(z-t)} - \frac{\Theta_3(2\Theta_1 - 1)}{(z-t)z} -$$

$$-\frac{2\Theta_1\Theta_2 - (2\Theta_\Sigma + 1)(\Theta_1 + \Theta_2) + \Theta_\Sigma(\Theta_\Sigma + 1)}{z(z-1)} -$$

$$-\frac{\Theta_1^2}{z^2} - \frac{\Theta_2^2}{(z-1)^2} - \frac{\Theta_3(\Theta_3 + 1)}{(z-t)^2}.$$

Constant monodromy, Schlesinger system.

A familly of the differential systems $d\Psi = A(z;t)dz\Psi$, has a fixed monodromy iff there is such 1-form $B(z,t)dt$ that $Adz + Bdt$ is flat. L. Schlesinger introduced an ansatz

$$A(z;t)dz + B(z,t)dt = \sum_k A^{(k)} \frac{dz - dz_k}{z - z_k} =: \omega,$$

$A^{(k)} = A^{(k)}(t), z_k = z_k(t)$.
The flatness condition $d\omega = \omega \wedge \omega$ is equivalent to the dynamical system $dA^{(k)} + [A^{(k)}, \sum_i \frac{dz_k - dz_i}{z_k - z_i}] = 0$. It is Hamiltonian on the Poisson space $gl^M(n, C)$ of the sets of matrices $\{A^{(k)}\} \in gl^M(n, C)$, a Hamiltonian form is $h = \sum_{i,j} \operatorname{tr} A^{(i)} A^{(j)} \frac{dz_i - dz_j}{z_i - z_j}$.

Normalized set of four matrices

The eigenvalues of $A^{(k)}$ are defined by the monodromy. The representative of the equivalence class of $\{A^{(k)}\}$ is:

$$A^{(1)} = \begin{pmatrix} \Theta_1' - pq & q \\ -p(pq - \Theta_1' + \Theta_1) & \Theta_1 + pq \end{pmatrix}, A^{(2)} = \begin{pmatrix} \Theta_2' & 1-q \\ 0 & \Theta_2 \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} \Theta_3' & 0 \\ a_{21}^{(3)} & \Theta_3 \end{pmatrix}, A^{(4)} = \begin{pmatrix} -\Sigma_{11} & -1 \\ -\Sigma_{11}\Sigma_{22} + \Theta_4'\Theta_4 & -\Sigma_{22} \end{pmatrix},$$

where Σ_{11}, Σ_{22} are the sums of the corresponding matrix elements:

$$\Sigma_{11} := -pq + \Theta_1' + \Theta_2' + \Theta_3', \Sigma_{22} := pq + \Theta_1 + \Theta_2 + \Theta_3,$$

$$a_{21}^{(3)} = p(pq - \Theta_1' + \Theta_1) - (pq - \sum_j \Theta_j' + \Theta_4')(pq + \sum_j \Theta_j - \Theta_4) - \Theta_4'\Theta_4.$$

Painlevé VI system.

The variables p, q are canonical on the symplectic quotient

$$\mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \mathcal{O}^{(3)} \times \mathcal{O}^{(4)} /\!\!/ \mathrm{GL}(2).$$

Let $z_1 = 0, z_2 = 1, z_3 = t, z_4 = \infty$. Then the Hamiltonian is

$$H := \frac{\operatorname{tr} A^{(3)} A^{(1)}}{t} + \frac{\operatorname{tr} A^{(3)} A^{(2)}}{t-1} = \frac{1}{t(t-1)} \operatorname{tr} A^{(3)}((t-1)A^{(1)} + tA^{(2)}) =$$

$$= \frac{q(q-1)(q-t)}{t(t-1)} \left(p^2 - p \left(\frac{\Theta_1''}{q} + \frac{\Theta_2''}{q-1} + \frac{\Theta_3''}{q-t} \right) \right) +$$

$$+ q \frac{\Theta_\Sigma''(\Theta_\Sigma'' - 2\Theta_4'')}{4t(t-1)} + *.$$

Where "$*$" does not influence the Hamiltonian equations.

Antiquantization.

Let us consider Heun equation with such normalization $\psi(z) \to \prod_k (z - z_k)^{\alpha_k} \psi(z)$ that one of two characteristical exponents at each finite point vanish:

$$\prod_{j=1}^{3}(z-z_j)\left(D^2\psi + \left(\sum_{j=1}^{3}\frac{1-\Theta_j}{z-z_j}\right)D\psi\right)+(\alpha\beta z-t(t-1)h)\psi = 0,$$

and compare it with the Hamiltonian system Painlevé VI:
$dp \wedge dq-$

$$-d\left(\prod_{j=1}^{3}(q-z_j)\left(p^2-\left(\sum_{j=1}^{3}\frac{\Theta_j''}{q-z_j}\right)p\right)+\widetilde{\Theta}_4 q\right)\wedge\frac{dt}{t(1-t)}.$$

Conclusion:

S. Yu. Slavyanov observed that the formal substitution

$$\{D = \frac{d}{dz}, z\} \to \{p, q\}$$

transforms the polynomial form of Heun equation with three zero charachteristical exponents into the Hamiltonian of the isomonodromic deformation of the Fuchsian system in the canonical variables.
He called it "the antiquantization". He treated h as energy and the location of the movable singularity as time.

The End.

Thank You!:)