q-Analogue of discriminant set and its computation

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Abstract. A generalization of the classical discriminant of the polynomial with arbitrary coefficients defined using the linear Hahn operator that decreases the degree of the polynomial by one is studied. The structure of the generalized discriminant set of the real polynomial, i.e., the set of values of the polynomial coefficients at which the polynomial and its Hahn operator image have a common root, is investigated. The structure of the generalized discriminant of the polynomial of degree n is described in terms of the partitions of n. Algorithms for the construction of a polynomial parameterization of the generalized discriminant set in the space of the polynomial coefficients are proposed. The main steps of these algorithms are implemented in a Maple library.

Introduction

Let $g : \mathbb{R} \to \mathbb{R} : x \mapsto g(x)$ be a given smooth one-to-one map of the real axis, which is the domain of polynomial $f(x)$ with arbitrary coefficients. We want to find conditions on the coefficients of the polynomial under which it has at least a pair of roots t_i, t_j satisfying the relation $g(t_i) = t_j$ and investigate the structure of the algebraic variety in the space of coefficients possessing such property.

Here we consider a generalization of the classical discriminant of the polynomial. This generalization naturally includes the classical discriminant and its analogs emerging when the q-differential and difference operators that have a well-developed calculus [1] and important applications [2] are used. It turned out that the constructs that were earlier obtained for investigating the discriminant [3] and resonance sets [4] can be extended for a more general case.

The aim of this research is to propose an efficient algorithm for calculating the parametric representation of all components of the g-discriminant set $\mathcal{D}_g(f)$ of the monic polynomial $f(x)$.

1. Generalized discriminant set

Definition 1. Define the q-**bracket** $[a]_q$, q-**Pochhammer symbol** $(a;q)_n$, q-**factorial** $[n]_q!$, q-**binomial coefficients (Gaussian) coefficients** $\begin{bmatrix} n \\ k \end{bmatrix}_q$ as follows:

$$[a]_q = \frac{q^a - 1}{q - 1}, a \in \mathbb{R}\backslash\{0\}, \quad (a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right), (a;q)_0 = 1,$$

$$[n]_q! = \prod_{k=1}^{n} [k]_q = \frac{(q;q)_n}{(1-q)^n}, q \neq 1, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!} = \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1}.$$

As $q \to 1$, all these objects become classical.

Define a g-analogue of the standard binomial $(x-a)^n$ so called g-binomial

$$\{x;t\}_{n;g} \equiv \prod_{i=0}^{n-1} (x - g^i(t)), \quad \{x;t\}_{0;q} = 1.$$

Here g^k is the k-th iteration of the diffeomorphism g, $k \in \mathbb{Z}$ (see below).

Let $f_n(x)$ be is a monic polynomial of degree n with complex coefficients defined by

$$f_n(x) \stackrel{\text{def}}{=} x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n.$$

Let $\mathbb{P}$ be the space of polynomials over $\mathbb{R}$ and let g

$$g : \mathbb{R} \to \mathbb{R} : x \mapsto qx + \omega, \quad q, \omega \in \mathbb{R}, \quad q \neq \{-1, 0\},$$

be a linear diffeomorphism on $\mathbb{R}$ that induces a linear **Hahn operator** $\mathcal{A}_{\mathrm{g}}$ on $\mathbb{P}$, satisfying the following two conditions:

1. the degree reduction: $\deg(\mathcal{A}_{\mathrm{g}}\, f_n)(x) = n - 1$; in particular, $\mathcal{A}_{\mathrm{g}}\, x = 1$;
2. Leibnitz rule analogue:

$$(\mathcal{A}_{\mathrm{g}}\, x f_n)(x) = f_n(x) + g(x)(\mathcal{A}_{\mathrm{g}}\, f_n)(x).$$

The Hahn operator $\mathcal{A}_{\mathrm{g}}$ called below g-**derivative** has the form

$$(\mathcal{A}_{\mathrm{g}}\, f)(x) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(qx+\omega) - f(x)}{(q-1)x + \omega}, & x \neq \omega_0, \\ f'(\omega_0), & x = \omega_0, \end{cases} \tag{1}$$

where $\omega_0 = \omega/(1-q)$ is the fixed point of g. Parameters q and ω are satisfied the following conditions $q, \omega \in \mathbb{R}$, $q \neq \{-1, 0\}$ and $(q, \omega) \neq (1, 0)$. The g-derivative $\mathcal{A}_{\mathrm{g}}$ can be considered as a generalization of the q-differential Jackson operator $\mathcal{A}_q$ at $\omega = 0$, $q \neq 1$, as the difference operator Δ_ω at $q = 1$ and as the classical derivative d/dx in the limit $q \to 1$ and $\omega = 0$.

q-Analogs of many mathematical objects emerged already in Euler's works, and then were elaborated by many mathematicians (see the historical review in [1]). The q-calculus has recently became a part of the more general construct called quantum calculus [5]. It has numerous applications in various fields of modern mathematics and theoretical physics. For example, many applications related to the theory of orthogonal polynomials and their various generalizations, it is important to determine the conditions on the coefficients a_i, $i = 1, \ldots, n$, of the polynomial $f_n(x)$ under which it has roots satisfying $g(t_i) = t_j$.

Definition 2. The pair of roots t_i, t_j, $i, j = 1, \ldots, n$, $i \neq j$ of the polynomial $f_n(x)$ is said to be **g-coupled** if $g(t_i) = t_j$.

Let consider the following problem.

Problem. In the coefficient space $\Pi \equiv \mathbb{C}^n$ of the polynomial $f_n(x)$, investigate the **g-discriminant set** denoted $\mathcal{D}_g(f_n)$ on which this polynomial has at least one pair of g-coupled roots.

Definition 3. The sequence $\mathrm{Seq}_g^{(k)}(t_1)$ of g-coupled roots of length k is defined as the finite sequence $\{t_i\}$, $i = 1, \ldots, k$ in which each term, beginning with the second one, is a g-coupled root of the preceding term: $g(t_i) = t_{i+1}$. The initial root t_1 is called the generating root of the sequence $\mathrm{Seq}_g^{(k)}(t_1)$.

For each fixed set of parameters q, ω, the g-discriminant set $\mathcal{D}_g(f_n)$ consists of a finite set of varieties $\mathcal{V}_k$ on each of which $f_n(x)$ has k sequences $\mathrm{Seq}_g^{(l_i)}(t_i)$ of g-coupled roots of length $_i$ with different generating roots t_i, $i = 1, \ldots, k$. To obtain an expression for the generalized (sub)discriminant of the polynomial $f_n(x)$ in terms of its coefficients, any method available in the classical elimination theory can be used. If we replace the derivative $f_n'(x)$ by the polynomial $\mathcal{A}_{\mathrm{g}}\, f_n(x)$, then any matrix method for calculating the resultant of a pair of polynomials gives an expression of the generalized k-th subdiscriminant $D_g^{(k)}(f_n)$ (see [6, 3] for details).

Theorem 1. *The polynomial $f_n(x)$ has exactly $n-d$ different sequences of g-coupled roots, iff the first nonzero element in the sequence of i-th generalized subdiscriminants $D_g^{(i)}(f_n)$ is the subdiscriminat $D_g^{(d)}(f_n)$ with the index d.*

2. Algorithm of parametrization of $\mathcal{D}_g(f_n)$ and its implementation

Definition 4. The partition λ of a natural number n is any finite nondecreasing sequence of natural numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$, for which $\sum_{i=1}^{k} \lambda_i = n$. Each partition λ will be written as $\lambda = [1^{n_1} 2^{n_2} 3^{n_3} \ldots]$.

Consider the partition $\lambda = [1^{n_1} 2^{n_2} 3^{n_3} \ldots]$ of the natural number n. The quantity i in the partition λ determines the length of the sequence of g-coupled roots for the corresponding generating root t_i, and n_i is the number of different generating roots determining the sequence of roots of length i. Every partition λ of n determines the structure of the g-coupled roots of the polynomial $f_n(x)$, and this structure is associated with the algebraic variety $\mathcal{V}_l^i$, $i = 1, \ldots, p_l(n)$ of dimension l corresponding to the number of different generating roots t_i in the coefficient space Π.

Consider the partition $\left[n^1\right]$ corresponding to the case when there is a unique sequence of roots of length n specified by the generating root t_1. Then, the polynomial $f_n(x)$ is a g-binomial $\{x;t_1\}_{n;g}$ and its coefficients a_i can be represented in terms of the elementary symmetric polynomials $\sigma_i(x_1, x_2, \dots, x_n)$ calculated on the roots $g^j(t_1)$, $j = 0, \dots, n-1$,

$$a_i = (-1)^i \sigma_i\left(t_1, g(t_1), \dots, g^{n-1}(t_1)\right), \; i = 1, \dots, n.$$

Let consider the polynomial $f_n(x) \equiv \{x;t_1\}_{n;g}$ with the structure of roots corresponding to the partition $\left[n^1\right]$. Using [7, Lemmas 2, 3], we conclude that, for every k such that $0 < k \le n$, it holds that

$$\sum_{i=0}^{k} \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{[n-i]_q!}{[n]_q!} \left(\mathcal{A}_{\mathrm{g}}{}^i f_n\right)(t_1)\{t_2;t_1\}_{i;g} = f_k(x;t_2) \cdot f_{n-k}(x; g^k(t_1)), \tag{2}$$

where $(\mathcal{A}_{\mathrm{g}}{}^0 f)(x) \equiv f(x)$. Therefore, formula (2) allows us to pass from the polynomial with the structure of roots corresponding to the partition $\left[n^1\right]$ to a polynomial with the structure of roots determined by the partitions $\left[k^1(n-k)^1\right]$ or $\left[(n/2)^2\right]$, if $k = n/2$.

Theorem 2. *Let there be a variety $\mathcal{V}_l$, $\dim \mathcal{V}_l = l$ on which the polynomial $f_n(x)$ has different sequences of g-coupled roots and the sequence of roots $\mathrm{Seq}_g^{(m)}(t_1)$ has length $m > 1$. The roots of the other sequences are not g-coupled with all roots of the sequence $\mathrm{Seq}_g^{(m)}(t_1)$. Let $\mathbf{r}_l(t_1, \dots, t_l)$ be a parameterization of the variety $\mathcal{V}_l$. Then for $0 < k < n$, the formula*

$$\mathbf{r}_l(t_1, \dots, t_l, t_{l+1}) = \mathbf{r}_l(t_1, \dots, t_l) + \sum_{i=1}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{[m-i]_q!}{[m]_q!} \left(\mathcal{A}_{\mathrm{g}}{}^i \mathbf{r}_l\right)(t_1)\{t_{l+1};t_1\}_{i;g}$$

specify a parameterization of the part of $\mathcal{V}_{l+1}$ on which there are two sequences of roots $\mathrm{Seq}_g^{(m-k)}(g^k(t_1))$ and $\mathrm{Seq}_g^{(k)}(g(t_{l+1}))$, and the other sequences of roots are the same as on the original variety $\mathcal{V}_l$.

We introduce two basic operations that allow us to successively pass from the parametric representation of the one-dimensional variety $\mathcal{V}_1$ to the parameterization of all other components of the g-discriminant set $\mathcal{D}_g(f_n)$.

1. The operation of passing from the variety $\mathcal{V}_l$ to the variety $\mathcal{V}_{l+1}$ in Theorem 2 is called **ASCENT** of order k. If $f_n(x)$ has only real roots on this variety, then we obtain its complete parameterization; if there are complex roots, then we apply the following operation.
2. The operation called **CONTINUATION** makes it possible to obtain a parameterization of the entire variety $\mathcal{V}_{l+1}$ obtained by the **ASCENT** operation in the case when there are complex conjugate roots on it.

At each step of this algorithm, we remain within polynomial parameterizations; therefore, the following result holds.

Proposition. For fixed values of parameters (q, ω) of the Hahn operator (1), the g-discriminant set $\mathcal{D}_g(f_n)$ of the polynomial $f_n(x)$ admits a polynomial parameterization of each of the algebraic varieties $\mathcal{V}_l^k$, $l = 1, \dots, n-1$, $k = 1, \dots, p_l(n)$, that form this set.

For calculating the g-discriminant set $\mathcal{D}_g(f_n)$, a number of procedures in `Maple` and `Sympy` were developed. Their description and application to some examples are given in [8].

References

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