## $q$-Analogue of discriminant set and its computation

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## Outline

(1) Motivation and Definitions
(2) Computation of generalized discriminant
(3) Parametrization of the generalized discriminant set
(4) Example


#### Abstract

A generalization of the classical discriminant of a polynomial with arbitrary coefficients is defined using the linear Hahn operator that decreases the degree of the polynomial by one is studied. The structure of a generalized discriminant set of the polynomial, i.e., a set of values of the polynomial coefficients at which the polynomial and its Hahn operator image have at least one common root, is investigated $q$-analogue of classical elimination theory. The structure of the generalized discriminant of the polynomial of degree $n$ is described in terms of the partitions of $n$. Algorithms for the construction of a polynomial parametrization of the generalized discriminant set in the space of the polynomial coefficients are proposed. The main steps of these algorithms are implemented in a Maple library.


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## Motivation I

While considering stability of stationary point of a multiparameter Hamiltonian system ${ }^{a}$ one has
(1) to find the boundary of the stability set;
(2) to separate such domains of stability set where the Bruno's Theorem takes place.
> ${ }^{\text {a }}$ Batkhin A. B. [et al.]. Stability sets of multiparameter Hamiltonian systems. // Journal of Applied Mathematics and Mechanics. 2012. Vol. 76, no. 1. P. 56-92.

In both cases it is necessary to compute so called Resonance Set ${ }^{a}$ of the characteristic polynomial of matrix of a linearised Hamiltonian system.

[^0]
## Motivation II

Here we propose some generalization of classical discriminant of a polynomial, which naturally include its $q$-analogues which have a special kind of calculus ${ }^{a}$ and some important applications ${ }^{b}$.

${ }^{a}$ Kac V. [et al.]. Quantum Calculus. New York, Heidelber, Berlin : Springer-Verlag, 2002. 112 p. ; Ernst T. A Comprehensive Treatment of $q$-Calculus. Basel Heidelberg New York Dordrecht London: Springer-Verlag, 2012. 491 p. ; Aldwoah K. A. Generalized time scales and associated difference equations: PhD thesis / Aldwoah K. A. Cairo University, 2009 ; Brito da Cruz A. M. C. Symmetric Quantum Calculus: PhD thesis / Brito da Cruz Artur Migual C. Universidade de Aveiro, 2012.<br>${ }^{b}$ Gasper G. [и др.]. Basic Hypergeometric Series. 2nd. Cambridge : Cambridge University Press, 2004. 455 c. ; Koekoek R. [et al.]. Hypergeometric Orthogonal Polynomials and Their $q$-Analogues. Berlin Heidelberg : Springer-Verlag, 2010. 578 p..

## Some definitions I

Let $f_{n}(x)$ be a monic polynomial of degree $n$ with arbitrary coefficients

$$
f_{n}(x) \stackrel{\text { def }}{=} x^{n}+a_{1} x^{n-1}+a_{2} x^{n-2}+\cdots+a_{n}
$$

## Definition

The $n$-dimensional space $\Pi \equiv \mathbb{C}^{n}$ of its coefficients $a_{1}, a_{2}, \ldots a_{n}$ is called the coefficient space of polynomial $f_{n}(x)$.

Let $g$

$$
g: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto g(x)
$$

be a smooth one-to-one map of $\mathbb{R}$

## Some definitions II

Map $g$ induces a linear operator $\mathcal{A}$ over the space $\mathbb{P}$ of polynomials over $\mathbb{R}$, which satisfies two conditions:

1) order reducing: $\operatorname{deg}\left(\mathcal{A} f_{n}\right)(x)=n-1$;
2) Leibnitz rule: $\left(\mathcal{A} x f_{n}\right)(x)=f_{n}(x)+g(x)\left(\mathcal{A} f_{n}\right)(x)$.

Conditions 1) and 2) take place if $g$ satisfies the following

$$
\operatorname{deg}(g(x)+x)=1 \quad \text { and } \quad \operatorname{deg}\left(g^{2}(x)+x g(x)+x^{2}\right)=2 .
$$

## Hahn operator $\mathcal{A}_{g}$ I

Here and below the map $g$ is defined as follows

$$
g(x) \equiv q x+\omega, \quad q, \omega \in \mathbb{R}, \quad q \notin\{-1,0\} .
$$

$g(x)$ has stationary point $\omega_{0}=\omega /(1-q)$.

## Hahn operator $\mathcal{A}_{g}$ II

## Definition

Hahn operator ${ }^{3}$ on $\mathbb{P}$ is called below $g$-derivative:

$$
\left(\mathcal{A}_{g} f\right)(x) \stackrel{\text { def }}{=} \begin{cases}\frac{f(q x+\omega)-f(x)}{(q-1) x+\omega}, & x \neq \omega_{0} \\ f^{\prime}\left(\omega_{0}\right), & x=\omega_{0}\end{cases}
$$

${ }^{2}$ Hahn W. Über Orthogonalpolynome, die $q$-Differenzengleichungen genügen. // Mathematische Nachrichten. 1949. Jg. 2. S. 4-34.

## Hahn operator $\mathcal{A}_{g}$ III

Hahn operator $\mathcal{A}_{g}$ is a natural generalization of

- Jackson $q$-derivative

$$
\left(\mathcal{A}_{q} f\right)(x)=\frac{f(q x)-f(x)}{(q-1) x}
$$

for $\omega=0$ and $q \neq 1$;

- difference operator $\left(\Delta_{\omega} f\right)(x)=\frac{f(x+\omega)-f(x)}{\omega}$ for $q=1$;
- classical derivative $d / d x$ at the limit $q \rightarrow 1$ and $\omega=0$.


## Auxiliary definitions I

## Definition

$q$-bracket (or $q$-number) $[n]_{q}$ is defined as $[n]_{q}=\frac{q^{n}-1}{q-1}$, $q$-Pochhammer symbol ( $q$-shifted factorial) is defined as

$$
(a ; q)_{n}=\prod_{k=0}^{n-1}(1-a q)^{k}, \quad(a ; q)_{0}=1
$$

$q$-factorial $[n]_{q}$ ! is defined as $[n]_{q}!=\prod_{k=1}^{n}[k]_{q}=\frac{(q ; q)_{n}}{(1-q)^{n}}$.

## Auxiliary definitions II

## Definition

$q$-binomial (Gaussian) coefficients is defined as

$$
\left[\begin{array}{c}
n \\
k
\end{array}\right]_{q}=\frac{[n]_{q}!}{[n-k]_{q}![k]_{q}!}=\prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^{i}-1}
$$

## Definition

$g$-binom is

$$
\{x ; t\}_{n ; g} \stackrel{\text { def }}{=} \prod_{i=0}^{n-1}\left(x-g^{i}(t)\right), \quad\{x ; t\}_{0 ; g}=1
$$

## $g$-linked roots

## Definition

A pair of roots $t_{i}, t_{j}, i, j=1, \ldots, n, i \neq j$, of the polynomial $f_{n}(x)$ is called $g$-linked if $g\left(t_{i}\right)=t_{j}$ for $g(x)=q x+\omega$.

## Generalized discriminant set of polynomial $f_{n}(x)$

## Definition

$g$-discriminant set $\mathcal{D}_{g}\left(f_{n}\right)$ of the polynomial $f_{n}(x)$ is called the set of all points of the coefficient space $\Pi$ at which $f_{n}(x)$ has at least a pair of $g$-linked roots, i.e.

$$
\mathcal{D}_{g}\left(f_{n}\right)=\left\{P \in \Pi: \exists i, j=1, \ldots, n, g\left(t_{i}\right)=t_{j}\right\}
$$

## The goal of the talk

(1) to present an algorithm of constructing polynomial representation of the $g$-discriminant set $\mathcal{D}_{g}\left(f_{n}\right)$ of a polynomial $f_{n}(x)$

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## The goal of the talk

(1) to present an algorithm of constructing polynomial representation of the $g$-discriminant set $\mathcal{D}_{g}\left(f_{n}\right)$ of a polynomial $f_{n}(x)$
(2) to propose a constructive algorithm of computation of parametric representation of all components of $\mathcal{D}_{g}\left(f_{n}\right)$
(3) to demonstrate some applications of $g$-discriminant set

This work is a continuation and generalization of previously obtained results on the discriminant $\mathcal{D}\left(f_{n}\right)$ and resonance $\mathcal{R}_{q}\left(f_{n}\right)$ sets of polynomial $f_{n}(x)^{a}$. Some results were published in (Batkhin A. B. Parameterization of a Set Determined by the Generalized Discriminant of a Polynomial).

[^1]
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## Generalized discriminant

## Definition

Generalized discriminant $D(f ; \mathcal{A})$ is defined as follows

$$
D(f ; \mathcal{A}) \stackrel{\text { def }}{=}(-1)^{n(n-1) / 2} \operatorname{Res}_{x}(f(x),(\mathcal{A} f)(x))
$$

For Hahn operator $\mathcal{A}_{g}$ generalized discriminant is denoted $D_{g}(f)$.

## Sequence of roots

Polynomial $f_{n}(x)$ may have some sets of $g$-linked roots.

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## Definition

The sequence $\operatorname{Seq}_{g}^{(k)}\left(t_{1}\right)$ of $g$-linked roots of length $k$ (shortly sequence of roots) is called the finite part of a sequence $t_{i}, i=1, \ldots, k$, each member of which is a $g$-linked root of a previous one: $t_{i+1}=g\left(t_{i}\right), i=1, \ldots, k-1$. The value $t_{1}$ is called the generating root.

## GCD of a pair $f_{n}(x)$ and $\left(\mathcal{A}_{g} f_{n}\right)(x)$

The structure of roots of the greatest common divisors

$$
\tilde{f}_{g}(x) \stackrel{\text { def }}{=} \operatorname{gcd}\left(f_{n}(x),\left(\mathcal{A}_{g} f_{n}\right)(x)\right)
$$

completely describes all the sequence of roots $\operatorname{Seq}_{g}^{(k)}\left(t_{j}\right)$.

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## Statement

Let $d=\operatorname{deg} \tilde{f}_{g}>0$, then each sequence of roots $\operatorname{Seq}_{g}^{(k)}\left(t_{j}\right), k \leqslant d$ of polynomial $\tilde{f}_{g}(x)$ corresponds to sequence of roots $\operatorname{Seq}_{g}^{(k+1)}\left(t_{j}\right)$ of polynomial $f_{n}(x)$.

## Subresultant computation methods ${ }^{1}$

## Matrix methods

are based on building an auxiliary matrix the elements of which are calculated given the coefficients of the pair of polynomials. The minors of this matrix taken in a certain order are subresultants of the corresponding orders.

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## The methods based on Euclid's algorithm and pseudo-division (K. Jacobi, 1836)

are iterative. They considerably reduce the computations and provide the subresultant of the corresponding order at each step. The majority of CAS use these methods in their built-in procedures for subresultant calculations.

## Sylvester's method I

## Definition

Sylvester's matrix $\operatorname{Sylv}(f, g)$ of pair of polynomials $f(x)$ and $g(x)$ for which $n=\operatorname{deg} f(x)$ and $m=\operatorname{deg} g(x)$, is defined as the $(n+m) \times(n+m)$, square matrix in which the rows are the vectors composed of the coefficients of the polynomials

$$
\begin{aligned}
& x^{m-1} f(x), x^{m-2} f(x), \ldots, \\
& \\
& x f(x), f(x), g(x), x g(x), \ldots, x^{n-2} g(x), x^{n-1} g(x)
\end{aligned}
$$

in the basis $x^{n+m-1}, \ldots, x, 1$.

## $k$-th generalized subdiscriminant

## Definition

Let $\mathbf{M}_{n}$ be a square matrix of size $n \times n$. Then, the matrix $\mathbf{M}_{n-k}, k<[n / 2]$, obtained by taking out the $k$ extreme rows and columns of the original matrix $\mathbf{M}_{n}$ is called its $k$-th inner.

## Definition

The determinant of the $k$-th inner of Sylvester's matrix $\operatorname{Sylv}(f, g)$ is called the $k$-th subresultant $\operatorname{Res}_{x}^{(k)}(f, g)$ of a pairs of $f(x)$ and $g(x)$.
Let call the $k$-th generalized subdiscriminant $D_{g}^{(k)}\left(f_{n}\right)$ of the polynomial $f_{n}(x)$ the determinant of $k$-th inner of generalized Sylvester matrix $\operatorname{Sylv}_{g}\left(f_{n}\right) \stackrel{\text { def }}{=} \operatorname{Sylv}\left(f_{n}(x),\left(\mathcal{A}_{g} f_{n}\right)(x)\right)$.

Example 1: $D_{g}^{(k)}\left(f_{3}\right)$ I

Cubic polynomial

$$
f_{3}=x^{3}+a_{1} x^{2}+a_{2} x+a_{3}
$$

has $g$-derivative

$$
\left(\mathcal{A}_{g} f_{3}\right)(x)=[3]_{q} x^{2}+\left([2]_{q} a_{1}+(2 q+1) \omega\right) x+\omega^{2}+\omega a_{1}+a_{2}
$$

$$
\operatorname{Sylv}_{g}\left(f_{3}\right) \equiv\left(\begin{array}{ccccc}
1 & a_{1} & a_{2} & a_{3} & 0 \\
0 & 1 & a_{1} & a_{2} & a_{3} \\
0 & 0 & a_{0}^{\prime} & a_{1}^{\prime} & a_{2}^{\prime} \\
0 & a_{0}^{\prime} & a_{1}^{\prime} & a_{2}^{\prime} & 0 \\
a_{0}^{\prime} & a_{1}^{\prime} & a_{2}^{\prime} & 0 & 0
\end{array}\right),
$$

where $a_{i}^{\prime}, i=0,1,2$ are corresponding coefficients of $g$-derivative of $f_{n}(x)$.

Example 1: $D_{g}^{(k)}\left(f_{3}\right)$ II

Cubic $f_{3}(x)$ has following generalized subdiscriminants

$$
\begin{aligned}
& D_{g}^{(2)}\left(f_{3}\right)=[3]_{q}, \\
& D_{g}^{(1)}\left(f_{3}\right)=q a_{1}^{2}+2 \omega(q-1) a_{1}-[3]_{q} a_{2}-3 \omega^{2}, \\
& D_{g}^{(0)}\left(f_{3}\right)=-\omega^{2} q^{2} a_{1}^{4}+\omega q^{2}(q-1) a_{1}^{3} a_{2}-q^{2}[2]_{q}^{2} a_{1}^{3} a_{3}-2 \omega^{3} q(q-1) a_{1}^{3}+ \\
& \quad+q^{3} a_{1}^{2} a_{2}^{2}+3 \omega^{2} q\left(q^{2}+1\right) a_{1}^{2} a_{2}-\omega q(q-1)(2 q+1)(q+2) a_{1}^{2} a_{3}- \\
& \quad-\omega^{4}\left(q^{2}-4 q+1\right) a_{1}^{2}-\omega q(q-1)[3]_{q} a_{1} a_{2}^{2}+ \\
& \quad+q[3]_{q}\left(q^{2}+4 q+1\right) a_{1} a_{2} a_{3}+\omega^{3}(q-1)\left(2 q^{2}+q+2\right) a_{1} a_{2}- \\
& \quad-\omega^{2}(q+2)(2 q+1)(q-1)^{2} a_{1} a_{3}+2 \omega^{5}(q-1) a_{1}-q^{2}[2]_{q}^{2} a_{2}^{3}- \\
& \quad-\omega^{2}[3]_{q}^{2} a_{2}^{2}+\omega(q-1)(2 q+1)(q+2)[3]_{q} a_{2} a_{3}-2 \omega^{4}[3]_{q} a_{2}- \\
& \quad-[3]_{q}^{3} a_{3}^{2}+\omega^{3}(q-1)(2 q+1)(q+2) a_{3}-\omega^{6} .
\end{aligned}
$$

## Theorem 1

Polynomial $f_{n}(x)$ has exactly $n-d$ different sequences of roots $\operatorname{Seq}_{g}{ }_{g}^{(i)}\left(t_{j}\right)$, $j=1, \ldots, n-d$ if and only if in the sequence $\left\{D_{g}^{(i)}\left(f_{n}\right), i=0, \ldots, n-1\right\}$ of $i$-th generalized subdiscriminants of $f_{n}(x)$ the first nonzero subdiscriminant is $d$-th generalized subdiscriminant $D_{g}^{(d)}\left(f_{n}\right)^{\text {a }}$.

[^2]
## GCD computation

One can obtain an expression of $\tilde{f}(x)$, which is GCD of a pair $f_{n}(x)$ and $\left(\mathcal{A}_{g} f\right)(x)$. Let denote by $\mathbf{M}_{d}^{(i)}$ the $d$-th inner of modified Sylvester matrix $\operatorname{Sylv}_{g}\left(f_{n}\right)$ in which $2 n-d$ - 1-th column is replaced with $2 n-d+i-1$-th column, and $M_{d}^{(i)}$ its determinant.

## Statement

If the first non-zero subdiscriminant $D_{g}^{(i)}\left(f_{n}\right)$ has number $d$, then

$$
\operatorname{gcd}\left(f_{n}(x),\left(\mathcal{A}_{q, \omega} f_{n}\right)(x)\right)=D_{g}^{(d)} x^{d}+M_{d}^{(1)} x^{d-1}+\cdots+M_{d}^{(i)}
$$

## Example 2: $g$-linked roots of $f_{3}$ I

Let first non-zero subdiscriminant has number 1: $D_{g}^{(1)}\left(f_{3}\right) \neq 0$. So, $g$-linked root $t_{1}$ is a root of polynomial $\tilde{f}_{3} \equiv D_{g}^{(1)}\left(f_{3}\right) x+M_{1}^{(1)}\left(f_{3}\right)$, where

$$
\begin{array}{r}
M_{1}^{(1)}\left(f_{3}\right)=\omega q^{2} a_{1}^{2}+q^{2} a_{1} a_{2}+\omega^{2}\left(q^{2}-2 q-1\right) a_{1}-\omega(2 q+1) a_{2}- \\
-[3]_{q}^{2} a_{3}-\omega^{3}(2 q+1)
\end{array}
$$

## Example 2: $g$-linked roots of $f_{3}$ II

Let $D_{g}^{(0)}\left(f_{3}\right)=D_{g}^{(1)}\left(f_{3}\right)=0$, then $\widetilde{f_{3}}$ is a quadratic polynomial with a pair of $g$-linked roots:

$$
\tilde{f}_{3} \equiv[3]_{q} x^{2}+\left((q+1) a_{1}+(2 q+1) \omega\right) x+\omega^{2}+\omega a_{1}+a_{2} .
$$

Generating root $t_{1}$ is:

$$
t_{1}=-\frac{q a_{1}+(q+2) \omega}{[3]_{q}!} .
$$

In this case cubic $f_{3}(x)$ is a $g$-binomial $f_{3}(x) \equiv\left\{x, t_{1}\right\}_{3 ; g}$.

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## Structure of $\mathcal{D}_{g}\left(f_{n}\right)$

Generalized discriminant set $\mathcal{D}_{g}\left(f_{n}\right)$ consists of the finite number of varieties $\mathcal{V}_{l}, \operatorname{dim} \mathcal{V}_{l}=l$, on each of which polynomial $f_{n}(x)$ has $l$ sequences of roots with different generating roots. Total length of these sequences is equal to the degree of polynomial.

Total number of such varieties $\mathcal{V}_{l}$ depends on the number of partitions $p(n)$ of natural number $n$.

## Partitions of $n$

## Definition

A partition $\lambda$ of a natural $n$ is any finite non-decreasing sequence of natural numbers $\lambda_{1} \leqslant \lambda_{2} \leqslant \cdots \leqslant \lambda_{k}$, such that $\sum_{i=1}^{k} \lambda_{i}=n$. Each partition can be written in the form

$$
\lambda=\left[1^{n_{1}} 2^{n_{2}} 3^{n_{3}} \ldots\right]
$$

where $n_{i}-$ is the number of repetitions $i$ in partition, i.e. $\sum_{i=1}^{k} i n_{i}=n$.

## Basic numerical functions related to partition

- Function $p(n)$ is the number of all partitions of $n$ (sequence A000041 $i n^{a}$ ).
- Function $p_{l}(n)$ is the number of partitions of $n$ into $l$ summands.
- Function $q(n)$ is the number of all partitions of $n$ into different summands (sequence A000009 in $^{b}$ ).
- Function $q_{l}(n)$ is the number of partitions of $n$ into $l$ different summands.

[^3]
## Connection between partitions and structure of $\mathcal{D}_{g}\left(f_{n}\right)$

Consider a partition

$$
\lambda=\left[1^{n_{1}} 2^{n_{2}} \ldots i^{n_{i}} \ldots\right]
$$

of $n \in \mathbb{N}$.

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of $n \in \mathbb{N}$.

- The value $i$ in the partition $\lambda$ defines the length of sequence $\operatorname{Seq}_{g}^{(i)}\left(t_{i}\right)$ for a corresponding generating root $t_{i}$;


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- The value $i$ in the partition $\lambda$ defines the length of sequence $\operatorname{Seq}_{g}^{(i)}\left(t_{i}\right)$ for a corresponding generating root $t_{i}$;
- The value $n_{i}$ defines the number of different generating roots, which give the sequences of root of the length $i$;


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- The value $i$ in the partition $\lambda$ defines the length of sequence $\operatorname{Seq}_{g}^{(i)}\left(t_{i}\right)$ for a corresponding generating root $t_{i}$;
- The value $n_{i}$ defines the number of different generating roots, which give the sequences of root of the length $i$;
- The value $l=\sum_{i} n_{i}$ is the number of different generating roots of the polynomial $f_{n}(x)$ for the certain values of the coefficient $q$ and $\omega$.

Number of varieties $V_{l} \subset \mathcal{D}_{g}\left(f_{n}\right)$

Any partition $\lambda$ of degree $n$ of polynomial $f_{n}(x)$ defines a certain structure of $q$-commensurable roots of this polynomial and it corresponds to some algebraic variety $\mathcal{V}_{l}^{i}, i=1, \ldots, p_{l}(n)$ of dimension $l$ in the coefficient space $\Pi$. The number of such varieties of dimension $l$ is equal to $p_{l}(n)$ and total number of all varieties consisting the resonance set $\mathcal{D}_{g}\left(f_{n}\right)$ is equal to $p(n)-$ 1. It is so because the partition $\left[1^{n}\right]$ corresponds to the case when all the $n$ roots of polynomial $f_{n}(x)$ are not commensurable.

## Main theorem

Algorithm for parametric representation of any variety $\mathcal{V}_{k}$ from the resonance set $\mathcal{D}_{g}\left(f_{n}\right)$ is based on the following

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Algorithm for parametric representation of any variety $\mathcal{V}_{k}$ from the resonance set $\mathcal{D}_{g}\left(f_{n}\right)$ is based on the following

## Theorem 2

Let $\mathcal{V}_{k}$, $\operatorname{dim} \mathcal{V}_{k}=k$, be a variety on which polynomial $f_{n}(x)$ has $k$ different sequences of $g$-linked roots and the sequences generated by the root $t_{1}$ has length $m>1$. Let denote by $\mathbf{r}_{k}\left(t_{1}, t_{2}, \ldots, t_{k}\right)$ parametrization of variety $\mathcal{V}_{k}$. The following formula

$$
\mathbf{r}_{k}\left(t_{1}, \ldots, t_{k+1}\right)=\mathbf{r}_{k}\left(t_{1}, \ldots, t_{k}\right)+\frac{t_{k+1}-t_{1}}{[m]_{q}}\left(\mathcal{A}_{g} \mathbf{r}_{k}\right)\left(t_{1}\right)
$$

gives parametrization of the part of variety $\mathcal{V}_{k+1}$, on which exists $\operatorname{Seq}_{q}^{(m-1)}\left(t_{1}\right)$, simple root $t_{k+1}$ and other roots are the same as on the initial variety $\mathcal{V}_{k}{ }^{a}$.

[^4]
## Geometrical interpretation

From the geometrical point of view Theorem 2 means that part of variety $\mathcal{V}_{q+1}$ is formed as a ruled hypersurface by the secant lines, which cross its directrix $\mathcal{V}_{q}$ at two points defined by such values of parameters $t_{1}^{\prime}$ and $t_{1}^{\prime \prime}$ that $g\left(t_{1}^{\prime}\right)=t_{1}^{\prime \prime}$. At the limit $q \rightarrow 1$ mentioned above ruled surface becomes a tangent ruled surface which parametrization is

$$
\mathbf{r}_{k+1}=\mathbf{r}_{k}+\frac{t_{k+1}-t_{1}}{m} \cdot \frac{\partial \mathbf{r}_{k}}{\partial t_{1}}
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$$

If polynomial $f_{n}(x)$ has on the variety $\mathcal{V}_{q+1}$ pairs of complex-conjugate roots it is necessary to make continuation of obtained parametrization (1).

Hierarchial structure of $\mathcal{D}_{g}\left(f_{n}\right)$

Let start from partition $\left[n^{1}\right]$ which corresponds to variety $\mathcal{V}_{1}$ with the only sequence $\operatorname{Seq}_{g}^{(n)}\left(t_{1}\right)$ of roots on it.
Parametrization of $\mathcal{V}_{1}$ can be expressed by means of elementary symmetric polynomials $\sigma_{i}$ computed on the sequence of $g$-linked roots $\left\{t_{1}, g\left(t_{1}\right), \ldots, g^{n-1}\left(t_{1}\right)\right\}$ :

$$
a_{i}=(-1)^{i} \sigma_{i}\left(t_{1}, g\left(t_{1}\right), \ldots, g^{n-1}\left(t_{1}\right)\right), \quad i=1, \ldots, n .
$$

One can apply transformation (1) of the Theorem 2 in succession and finally can obtain parametrization of variety $\mathcal{V}_{n-1}$ of the maximal dimension $\operatorname{dim} \mathcal{V}_{n-1}=n-1$. There exists only one sequence of roots of the length 2 on it and other roots are simple.

## Example 3: Parametrization of $\mathcal{D}_{g}\left(f_{3}\right)$ I

Generalized discriminant set $\mathcal{D}_{g}\left(f_{3}\right)$ consists of two varieties

$$
\begin{aligned}
& \mathcal{V}_{1}:\left\{a_{1}=-\left(1+q+q^{2}\right) t_{1}, a_{2}=q\left(1+q+q^{2}\right) t_{1}^{2}, t_{3}=-q^{3} t_{1}^{3}\right\} \\
& \mathcal{V}_{2}:\left\{a_{1}=-(1+q) t_{1}-t_{2}, a_{2}=q t_{1}^{2}+(1+q) t_{1} t_{2}, a_{3}=-q t_{1}^{2} t_{2}\right\}
\end{aligned}
$$

which corresponds to partitions $\left[3^{1}\right]$ and $\left[1^{1} 2^{1}\right], \mathcal{V}_{1} \subset \mathcal{V}_{2}$.

From the geometrical point of view the $\mathcal{D}_{g}\left(f_{3}\right)$ is a ruled surface $\mathcal{V}_{2}$ with screwed spatial cubic curve $\mathcal{V}_{1}$ as a directrix.

## Example 3: Parametrization of $\mathcal{D}_{g}\left(f_{3}\right)$ II



## Example 3: Parametrization of $\mathcal{D}_{g}\left(f_{3}\right)$ III



## More general construction

## Theorem 3

Let $\mathcal{V}_{k}, \operatorname{dim} \mathcal{V}_{k}=k$, be a variety on which polynomial $f_{n}(x)$ has $k$ different sequences of $g$-linked roots and the sequence generated by the root $t_{1}$ has length $m>1$. Let denote by $\mathbf{r}_{k}\left(t_{1}, t_{2}, \ldots, t_{k}\right)$ parametrization of variety $\mathcal{V}_{k}$. The following formula

$$
\begin{align*}
& \mathbf{r}_{l}\left(t_{1}, \ldots, t_{l}, t_{l+1}\right)=\mathbf{r}_{l}\left(t_{1}, \ldots, t_{l}\right)+ \\
&  \tag{1}\\
& \quad+\sum_{i=1}^{k}\binom{k}{i}_{q} \frac{[m-i]_{q}!}{[m]_{q}!}\left(\mathcal{A}_{g}^{i} \mathbf{r}_{l}\right)\left(t_{1}\right)\left\{t_{l+1} ; t_{1}\right\}_{i ; g}
\end{align*}
$$

gives parametrization of the part of variety $\mathcal{V}_{k+1}$, on which exist two sequences of $g$-linked roots $\operatorname{Seq}_{g}^{(m-k)}\left(g^{k}\left(t_{1}\right)\right)$ and $\operatorname{Seq}_{g}^{(k)}\left(g\left(t_{l+1}\right)\right)$ and other roots are the same as on the initial variety $\mathcal{V}_{k}{ }^{a}$.

[^5]
## Two operations

Let define the following two operations, which make it possible to obtain parametrization of each variety $\mathcal{V}_{l}$ of dimensions from 2 to $n-1$.
"ASCENT" allows to pass from variety $\mathcal{V}_{i}$ to the part of another variety $\mathcal{V}_{i+1}$ with dimension one greater by (1) from Theorem 3.

## Two operations

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"CONTINUATION" allows to get the parametrization of the whole variety $\mathcal{V}_{i+1}$ obtained on the previous step.

One can combine successively mentioned above operations to obtain parametric representation of each variety $\mathcal{V}_{i}$ of the generalized discriminant set $\mathcal{D}_{g}\left(f_{n}\right)$.

## Software implementation

Maple library gDiscrSet was implemented, which includes the set of procedures:

- gSubDiscrim for computation of generalized discriminant;
- MkParV1 for parametric representation of variety $\mathcal{V}_{1}$;
- ProcUp implements operation "ASCENT";
- ProcCont implements operation"CONTINUATION".
- and some auxiliary procedures.

Now implementation for SymPy is under construction.

## Main result

## Statement

Generalized discriminant set $\mathcal{D}_{g}\left(f_{n}\right)$ of real polynomial $f_{n}(x)$ for a certain value of parameters $q$ and $\omega$ allows polynomial parametrization of each its variety $\mathcal{V}_{k} \subset \mathcal{D}_{g}\left(f_{n}\right)$, which can be effectively computed by means of classical technic of elimination theory.

## Outline

## (1) Motivation and Definitions

(2) Computation of generalized discriminant
(3) Parametrization of the generalized discriminant set
(4) Example

Double pendulum with tracking force I


## Double pendulum with tracking force II

$$
\chi(\lambda) \stackrel{\text { def }}{=} \lambda^{4}+\frac{1}{2}\left(\gamma_{1}+6 \gamma_{2}\right) \lambda^{3}+\frac{1}{2}\left(\gamma_{1} \gamma_{2}-2 p+7\right) \lambda^{2}+\frac{1}{2}\left(\gamma_{1}+\gamma_{2}\right) \lambda+\frac{1}{2},
$$

where $\gamma_{i}$ are some parameters ${ }^{a}$.
${ }^{\text {a }}$ Herrmann $G$. [et al.]. On the destabilizing effect of damping in nonconservative elastic systems. // Trans. AMSE, J. Appl. Mech. 1965.
Vol. 32. P. 592-597.

## Problem

Describe $q$-discriminant set $\mathcal{D}_{q}(\chi)$, i.e. set in coefficient space on which polynomial $\chi(\lambda)$ has at least a pair of $q$-commensurable roots: $\lambda_{j}=q \lambda_{i}$, $i \neq j, q \notin\{-1,0\}$.

## Double pendulum with tracking force III

Split the problem into 2 parts:

- describe the set $\mathcal{D}_{q}\left(f_{4}\right)$ of quartic

$$
f_{4}(x) \stackrel{\text { def }}{=} x^{4}+a_{1} x^{3}+a_{2} x^{2}+a_{3} x+\frac{1}{2}
$$

- provide transformation $S$ from coefficients $a_{1}, a_{2}, a_{3}$ to parameters $\gamma_{1}, \gamma_{2}, p$ :

$$
\begin{aligned}
\gamma_{1} & =\frac{2}{5}\left(6 a_{3}-a_{1}\right), \gamma_{2}=\frac{2}{5}\left(a_{1}-a_{3}\right) \\
p & =-\frac{2}{25}\left(a_{1}^{2}-7 a_{1} a_{3}+6 a_{3}^{2}\right)-a_{2}+\frac{7}{2}
\end{aligned}
$$

## Computation scheme for quartic $\chi(\lambda)$



Figure 1: Scheme of computation of varieties $\mathcal{V}_{k}$ of discriminant $\mathcal{D}_{g}(\chi)$. Procedure "ASCENT" denoted Up, procedure "CONTINUATION" by green circle.

## Parametrization of $\mathcal{D}_{q}(\chi)$ I

$$
\begin{gathered}
V_{0}:\left\{a_{1}= \pm \frac{[2]_{q}\left(q^{2}+1\right)}{2^{1 / 4} q^{3 / 2}}, \quad a_{2}=\frac{[3]_{q}\left(q^{2}+1\right)}{q^{2} \sqrt{2}}, \quad a_{3}=\frac{a_{1}}{\sqrt{2}}\right\} \\
\mathcal{V}_{1}^{1}\left(f_{4}\right):\left\{a_{1}=-[3]_{q} t_{1}-\frac{1}{2\left(q t_{1}\right)^{3}}, a_{2}=[3]_{q}\left(q t_{1}^{2}+\frac{1}{2 q^{3} t_{1}^{2}}\right)\right. \\
\left.a_{3}=-\left(q t_{1}\right)^{3}-\frac{[3]_{q}}{2 q^{2} t_{1}}\right\} .
\end{gathered}
$$

## Parametrization of $\mathcal{D}_{q}(\chi)$ II

$$
\begin{aligned}
\mathcal{V}_{2}\left(f_{4}\right):\left\{\begin{aligned}
a_{1} & =-[2]_{q} t_{1}-2 t_{2}, \quad a_{2}=q t_{1}^{2}+2[2]_{q} t_{1} t_{2}+\frac{1}{2 q t_{1}^{2}} \\
a_{3} & \left.=-2 q t_{1}^{2} t_{2}-\frac{[2]_{q}}{2 q t_{1}}\right\}
\end{aligned}\right.
\end{aligned}
$$

## Parametrization of $\mathcal{D}_{q}(\chi)$ III

$\mathcal{V}_{1}^{2}\left(f_{4}\right)$ consists of three curves $\mathcal{L}_{2}^{ \pm}$и $\mathcal{L}_{3}$ :

$$
\begin{gathered}
\mathcal{L}_{2}^{ \pm}:\left\{a_{1}=-[2]_{q}\left(t_{1}+\frac{1}{\sqrt{2} q t_{1}}\right), a_{2}=q t_{1}^{2}+\frac{[2]_{q}^{2}}{\sqrt{2} q}+\frac{1}{2 q t_{1}^{2}},\right. \\
\left.a_{3}=-\frac{[2]_{q}}{\sqrt{2}}\left(t_{1}+\frac{1}{\sqrt{2} q t_{1}}\right)\right\}, \\
\mathcal{L}_{3}:\left\{a_{1}=-[2]_{q}\left(t_{1}-\frac{1}{\sqrt{2} q t_{1}}\right), a_{2}=q t_{1}^{2}-\frac{[2]_{q}^{2}}{\sqrt{2} q}+\frac{1}{2 q t_{1}^{2}},\right. \\
\left.a_{3}=-\frac{[2]_{q}}{\sqrt{2}}\left(t_{1}-\frac{1}{\sqrt{2} q t_{1}}\right)\right\} .
\end{gathered}
$$

## Parametrization of $\mathcal{D}_{q}(\chi)$ IV


1)

2)

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[^1]:    ${ }^{\text {a }}$ Batkhin A. B. Parameterization of the Discriminant Set of a Polynomial.

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