On the Pierce–Birkhoff conjecture and related problems.

Mark Spivakovsky

Let R be a real closed field and $B = R[x_1, \ldots, x_n]$ a polynomial ring over R in n variables.

Definition 0.1. *A function $g : R^n \to R$ is said to be* **piecewise polynomial** *if R^n can be covered by a finite collection of closed semi-algebraic sets P_i, $i \in \{1, \ldots, s\}$ such that for each i there exists a polynomial $g_i \in B$ satisfying $g|_{P_i} = g_i|_{P_i}$.*

Piecewise polynomial functions form a ring, containing B, which is denoted by $PW(B)$.

Consider the ring (contained in $PW(B)$) of all the functions obtained from B by iterating the operations of sup and inf. The Pierce–Birkhoff conjecture was stated by M. Henriksen and J. Isbell in the early nineteen sixties ([1] and [3]):

Conjecture 1. (Pierce-Birkhoff) *If $g : R^n \to R$ is in $PW(B)$, then there exists a finite family of polynomials $g_{ij} \in B$ such that $f = \sup_i \inf_j (g_{ij})$ (in other words, for all $x \in R^n$, $f(x) = \sup_i \inf_j (g_{ij}(x))$).*

In this talk, we will recall the definition of the real spectrum of a ring Σ, denoted by Sper Σ. In the nineteen eighties, generalizing the problem from the polynomial ring to an arbitrary ring Σ, J. Madden proved that the Pierce–Birkhoff conjecture for Σ is equivalent to a statement about an arbitrary pair of points $\alpha, \beta \in$ Sper Σ and their separating ideal $< \alpha, \beta >$; we refer to this statement as the **local Pierce-Birkhoff conjecture** at α, β. In [4] we introduced a stronger conjecture, also stated for a pair of points $\alpha, \beta \in$ Sper Σ and the separating ideal $< \alpha, \beta >$, called the **Connectedness conjecture**, about a finite family of elements $f_1, \ldots, f_r \in \Sigma$. In [6] we introduced a new conjecture, called the **Strong Connectedness conjecture**, and proved that the Strong Connectedness conjecture in dimension $n-1$ implies the strong connectedness conjecture in dimension n in the case when $ht(< \alpha, \beta >) \leq n-1$.

The Pierce-Birkhoff Conjecture for $r = 2$ is equivalent to the Connectedness Conjecture for $r = 1$; this conjecture is called the Separation Conjecture. The

Strong Connectedness Conjecture for $r = 1$ is called the Strong Separation Conjecture. In this talk fix a polynomial $f \in R[x, z]$ where $x = (x_1, \ldots, x_n), z$ are $n+1$ independent variables. We will define the notion of two points $\alpha, \beta \in \text{Sper } R[x, z]$ being in **good position** with respect to f. Our main result is a proof of the Strong Separation Conjecture in the case when α and β are in good position with respect to f. We also prove that, given a connected semi-algebraic set $D \subset R^n$, if the number of real roots of f, counted with or without multiplicity, is constant for all $x \in D$ then these roots are represented by continuous semi-algebraic functions $\phi_j : D \to R$.

References

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Mark Spivakovsky
CNRS, Institut de Mathématiques de Toulouse 118,
route de Narbonne, F-31062 Toulouse Cedex 9, France
e-mail: `mark.spivakovsky@gmail.com`