A New Approach to Effective Computation of the Dimension of an Algebraic Variety

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Abstract. We discuss a new method for computing the dimension of an algebraic variety. It is based on the effective version of the first Bertini theorem for hypersurfaces suggested by the author earlier.

Computation of the dimension of an algebraic variety is a classical problem in effective algebraic geometry. In the most simple case it is formulated as follows. Let k be a field with the algebraic closure $\overline{k}$. Given homogeneous polynomials $f_1, \ldots, f_m \in k[X_0, \ldots, X_n]$ the problem is to compute the dimension of the algebraic variety $\mathcal{Z}(f_1, \ldots, f_m)$ of all the common zeroes of the polynomials $f_1, \ldots, f_m$ in the projective space $\mathbb{P}^n(\overline{k})$.

Assume additionally that the degrees $\deg_{X_0,\ldots,X_n} f_j \leqslant d$ for an integer $d \geqslant 2$ for all $1 \leqslant i \leqslant m$. Then the number of coefficients of each polynomial f_j is at most $\binom{n+d}{n}$. So it is bounded from above by a polynomial in d^n.

On the other hand, one can verify whether the set $\mathcal{Z}(f_1, \ldots, f_m)$ is finite (or empty) and if $\#\mathcal{Z}(f_1, \ldots, f_m) < +\infty$ solve the homogeneous system $f_1 = \ldots = f_m = 0$ over the algebraically closed field $\overline{k}$. The complexity of this algorithm is polynomial in d^n and the size of the input data, see [4]. Actually the main ideas for solving homogeneous systems of polynomial equations with a finite number of roots are classical and were known at the beginning of the previous century, see [5].

Let us return to the general case. Now the probabilistic algorithm for computing the dimension of an algebraic variety is simple. Let s be an integer such that $-1 \leqslant s \leqslant n$. Let us choose linear forms $L_0, \ldots, L_s \in k[X_0, \ldots, X_n]$ randomly. Then the dimension $\dim \mathcal{Z}(f_1, \ldots, f_m)$ is the least s such that the set

$$\mathcal{Z}(f_1, \ldots, f_m, L_0, L_1, \ldots, L_s)$$

is empty. So one can compute the dimension of a projective algebraic variety probabilistically within the time polynomial in d^n and the size of the input data.

But to compute the dimension deterministically is much more difficult. In the case of arbitrary characteristic it is an open problem

(*) to construct a deterministic algorithm for computing the dimension of a projective algebraic variety $\mathcal{Z}(f_1,\dots,f_m)$ with bitwise complexity polynomial in d^n and the size of the input data.

We think that for arbitrary characteristic of the ground field this problem will not be solved in near future (say, in this century).

Still here there have been a major progress. In the case of the ground field of zero–characteristic we solved the problem (*), see [1]. We could obtain the main result of [1] using the methods of real algebraic geometry. After that we have developed the whole theory basing on these methods and get many important results. However, to many specialists it seemed unnatural to apply the methods of real algebraic geometry for varieties over algebraically closed fields. On the other hand, it is a fact that all other attempts to compute the dimension deterministically within the time polynomial in d^n and the size of the input data have been fruitless.

The situation has changed after the results of [2]. Namely, in [2] we got a very strong and explicit version of the first Bertini theorem for the case of a hypersurface. Now it is possible to attract the new ideas related to irreducibility and transversality of intersections of algebraic varieties. Quite probably (one should check the details) that in the case of the ground field of zero–characteristic one can solve the problem (*) with the help of [2] (and without using methods of real algebraic geometry).

These techniques are not sufficient for the case of the ground field of nonzero characteristic. Here the main difficulties are related to inseparability. But the situation is not so hopeless. In the case of nonzero characteristic one can use additionally the results [3]. We would like to formulate the following hypothesis.

(†) In the case on nonzero characteristic one can one construct a deterministic algorithm for computing the dimension of a projective algebraic variety $\mathcal{Z}(f_1,\dots,f_m)$ with bitwise complexity polynomial in $C(n)d^n$ and the size of the input data where the constant $C(n)$ depends only on n (more precisely, $C(n) < 2^{2^{n^C}}$ for an absolute constant $C > 0$, cf. [3]).

References

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