On a moduli space of the Wigner quasiprobability distributions

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Content

Context

Recently an ambiguity in specification of the Wigner quasiprobability distribution for finite-dimensional quantum system has been studied. It was shown that for an N-level quantum system one can construct $N-2$ parametric family of unitary non-equivalent Wigner quasiprobability distributions.

The main objective

In the report the moduli space of the Wigner quasiprobability distributions for N-dimensional quantum systems will be discussed and exemplified for a low dimensional cases, for a single qubit, qutrit and quatrit.

Introduction

Defining a quantum state

A state of an N-level quantum-mechanical system is described by a **density operator** ϱ acting on the $\mathbb{C}^N$ Hilbert space [a], satisfying the conditions:

$$\bullet\text{Hermicity}: \varrho^\dagger = \varrho\,; \qquad \bullet\text{Completeness}: tr(\varrho) = 1\,;$$

$$\bullet\text{Semi-positivity}: \langle\psi|\varrho|\psi\rangle \geq 0\,;$$

[a] The states with $\varrho^2 = \varrho$ are called pure states.

Physical quantities as quantum averages

The density operator ϱ determines the expectation value $\mathbb{E}(\hat{A})$ of a Hermitian operator $\hat{A}$ acting on $\mathcal{H}$: $\mathbb{E}(\hat{A}) = \mathrm{tr}\left(\hat{A}\,\varrho\right)$.

As statistical averages via probability distribution function (PDF) $\rho(q,p)$

For a function $A(q,p)$ defined over classical phase space Ω:

$$\mathbb{E}(A) = \int \mathrm{d}\Omega\; A(q,p)\,\rho(q,p)\,, \quad \text{with} \quad \int \mathrm{d}\Omega\; \rho(q,p) = 1\,.$$

QPDF as "quantum analogue" of the statistical PDF

The Wigner-Weyl transform

Invertible map between functions over the phase space and operators acting on the Hilbert space:

$$\hat{A} \rightleftarrows W_A(q,p)\,,$$

where $W_A(q,p)$ is called symbol of $\hat{A}$. For $\hat{A} = \varrho$ it is called **Wigner quasiprobability distribution function (QPDF)**.

The Wigner function

is constructed from the density matrix ϱ describing a quantum state and the **Stratonovich-Weyl self-dual kernel** $\Delta(\Omega_N)$ defined over the symplectic manifold Ω_N:

$$\boxed{W_\varrho(\Omega) = tr\left(\varrho\,\Delta(\Omega)\right)\,.}$$

Stratonovich-Weyl: operators($\mathcal{H}$) $\overset{\Delta(\Omega)}{\mapsto}$ functions(Ω), at that

1. *Reconstruction* of the state:

$$\varrho = \int_\Omega \mathrm{d}\Omega\, \Delta(\Omega)\, W_\varrho(\Omega)\,;$$

2. *Hermicity* of the kernel:

$$\Delta(\Omega) = \Delta(\Omega)^\dagger\,;$$

3. *Finite norm*:

$$tr\,(\varrho) = \int_\Omega \mathrm{d}\Omega\, W_\varrho(\Omega)\,, \qquad \int_\Omega \mathrm{d}\Omega\, \Delta(\Omega) = 1\,;$$

4. *Covariance*: the unitary symmetry $\varrho' = U(\alpha)\,\varrho\,U^\dagger(\alpha)$ induces the kernel transformation [a]

$$\Delta(\Omega') = U(\alpha)^\dagger\, \Delta(\Omega)\, U(\alpha)\,.$$

[a] Hence, the phase space measure is $SU(N)$ invariant Haar measure.

Deriving the "Master equations"

$$\varrho = Z_N^{-1} \int_{SU(N)} \mathrm{d}\mu_{SU(N)}\, \Delta(\Omega_N)\, \mathrm{tr}\,[\varrho\, \Delta(\Omega_N)]\ ,$$

$$\Big\downarrow\ \Delta(\Omega) = U(\Omega) P U^\dagger(\Omega)\,, \text{ where } \begin{array}{l} P = diag\|\pi_1, \cdots, \pi_N\| \\ \pi_1 \geq \pi_2 \geq \cdots \geq \pi_N \end{array}$$

$$Z_N^{-1} \int_{SU(N)} \mathrm{d}\mu_{SU(N)}\, (U\,P\,U^\dagger)_{ik}\, (U\,P\,U^\dagger)_{js}\, \varrho_{sj} = \varrho_{ik}\ ,$$

↓ 4th order Weingarten formula[1]

Standardisation condition[2]

Master equations:

$$tr\,(\Delta(\Omega)) = 1\,, \qquad tr\,\left(\Delta(\Omega)^2\right) = N\,.$$

[1] $\int d\mu U_{ab} U_{cd} U^\dagger_{ef} U^\dagger_{gh} = \frac{1}{N^2-1}\left(\delta_{af}\delta_{ch}\delta_{be}\delta_{dg} + \delta_{ah}\delta_{cf}\delta_{bg}\delta_{de}\right) - \frac{1}{N(N^2-1)}\left(\delta_{af}\delta_{ch}\delta_{bg}\delta_{de} + \delta_{ah}\delta_{cf}\delta_{be}\delta_{dg}\right)$.

[2] $Z_N^{-1} \int \mathrm{d}\mu_{SU(N)}\, W_A^{(\nu)}(\Omega) = tr\,(A)$.

The Stratonovich-Weyl kernel

$$\Delta(\Omega|\boldsymbol{\nu}) = \frac{1}{N} U(\Omega) \left[I + \kappa \sum_{\lambda \in H} \mu_s(\boldsymbol{\nu}) \lambda_s \right] U(\Omega)^\dagger , \quad \kappa = \sqrt{N(N^2-1)/2} ,$$

where

- H is the **Cartan subalgebra** in $SU(N)$,
- parameter $\boldsymbol{\nu} = (\nu_1 , \cdots , \nu_{N-2})$ labels members of the WF family,
- coefficients $\boxed{\sum_{s=2}^{N} \mu^2_{s^2-1}(\boldsymbol{\nu}) = 1}$.

A density matrix of an N-dimensional quantum system

$$\varrho_\xi = \frac{1}{N} \left[I + \sqrt{\frac{N\,(N-1)}{2}}\, (\boldsymbol{\xi}, \boldsymbol{\lambda}) \right] ,$$

where

- $\boldsymbol{\xi}$ is an (N^2-1)-dimensional Bloch vector,
- $\boldsymbol{\lambda} = \{\lambda_1 , \cdots , \lambda_{N^2-1}\}$ is $\mathfrak{su}(N)$ algebra basis.

A family of the Wigner functions

$$W_{\boldsymbol{\xi}}^{(\boldsymbol{\nu})}(\Omega_N) = \frac{1}{N}\left[1 + \frac{N^2-1}{\sqrt{N+1}}(\boldsymbol{n}, \xi)\right],$$

where

- $\boldsymbol{n} = \mu_3 \boldsymbol{n}^{(3)} + \cdots + \mu_{N^2-1}\boldsymbol{n}^{(N^2-1)}$,
- $\boldsymbol{n}^{(s^2-1)} = \frac{1}{2}\operatorname{tr}\left(U\lambda_{s^2-1}U^\dagger\lambda_\mu\right), \quad s = \overline{2, N}$.

The spectrum $\{\pi_1, \cdots, \pi_N\}$ of the Stratonovich-Weyl kernel:

$$\pi_i = \frac{1}{N}\left(1 + \sqrt{2}\kappa \sum_{s=i+1}^{N} \frac{\mu_{s^2-1}}{\sqrt{s(s-1)}} - \kappa\sqrt{\frac{2(i-1)}{i}}\mu_{i^2-1}\right).$$

Constraints on the spherical angles

The spherical $(N-2)$ angles:

$$\begin{aligned}
\mu_3 &= \sin\psi_1 \cdots \sin\psi_{N-2}\,, \\
&\vdots \\
\mu_{i^2-1} &= \sin\psi_1 \cdots \sin\psi_{N-i}\cos\psi_{N-i+1}\,, \\
&\vdots \\
\mu_{N^2-1} &= \cos\psi_1\,, \qquad i = \overline{2, N}\,.
\end{aligned}$$

For decreasing order $\pi_1 \geq \cdots \geq \pi_N$

$$\boxed{\mu_3 \geq 0\,, \qquad \mu_{(i+1)^2-1} \geq \sqrt{\frac{i-1}{i+1}}\,\mu_{i^2-1}\,, \quad i = \overline{2, N-1}\,.}$$

Examples: qubit, qutrit and quatrit

The Wigner function of a single qubit

A generic **qubit** quantum state is parameterized in a standard way

$$\varrho_{qubit} = \frac{1}{2}\left(I + \boldsymbol{r}\cdot\boldsymbol{\sigma}\right)$$

by the Bloch vector $\boldsymbol{r} = (r\sin\psi\cos\phi\,,\; r\sin\phi\sin\phi\,,\; r\cos\psi)\,.$
The master equations determine the spectrum:

$$\operatorname{spec}\left(P^{(2)}\right) = \left\{\frac{1+\sqrt{3}}{2}\,,\;\frac{1-\sqrt{3}}{2}\right\}\,.$$

The **Wigner function** for a single qubit is

$$W_{\boldsymbol{r}}(\alpha,\beta) = \frac{1}{2} + \frac{\sqrt{3}}{2}\,(\boldsymbol{r},\boldsymbol{n})\;,$$

where $\boldsymbol{n} = (-\cos\alpha\sin\beta\,,\,\sin\alpha\sin\beta\,,\,\cos\beta)$ is the unit 3-vector.

Qutrit kernel and its fundamental region

A generic **qutrit** state is given by the density matrix

$$\varrho_{qutrit} = \frac{1}{3}\left(I + \sqrt{3}\sum_{\nu=1}^{8} \xi_\nu \lambda_\nu \right) .$$

The **Stratonovich-Weyl kernel**

$$\Delta(\Omega_3) = U(\Omega_3)\,\frac{1}{3}\left[I + 2\sqrt{3}\,(\mu_3\lambda_3 + \mu_8\lambda_8)\right]\, U(\Omega_3)^\dagger ,$$

where the coefficients

$$\mu_3(\nu) = \frac{\sqrt{3}}{4}\sqrt{(1+\nu)(5-3\nu)}\,, \quad \mu_8(\nu) = \frac{1}{4}(1-3\nu)$$

are functions of the parameter $\nu = \frac{1}{3} - \frac{4}{3}\cos(\zeta)$ with $\zeta \in [0,\ \pi/3]$ being the moduli parameter of the unitary nonequivalent WF of a qutrit.

The **Wigner function** of a single qutrit

$$W^{(\nu)}_{\boldsymbol{\xi}}(\Omega_3) = \frac{1}{3} + \frac{4}{3}\left[\mu_3\left(\boldsymbol{n}^{(3)}, \boldsymbol{\xi}\right) + \mu_8\left(\boldsymbol{n}^{(8)}, \boldsymbol{\xi}\right)\right],$$

with two orthogonal unit 8-vectors

$$n^{(3)}_\nu = \frac{1}{2}\mathrm{tr}\left[U\lambda_3 U^\dagger \lambda_\nu\right], \qquad n^{(8)}_\nu = \frac{1}{2}\mathrm{tr}\left[U\lambda_8 U^\dagger \lambda_\nu\right].$$

The **master equations**

$$tr\left(\Delta(\Omega)\right) = 1, \qquad tr\left(\Delta(\Omega)^2\right) = 3$$

determine one-parametric family of kernels $P^{(3)}(\nu)$.

One-parametric $P^{(3)}(\nu)$-family

- The spectrum of **generic** kernels:

$$\text{spec}\left(P^{(3)}(\nu)\right) = \left\{\frac{1-\nu+\delta}{2}, \frac{1-\nu-\delta}{2}, \nu\right\},$$

where $\delta = \sqrt{(1+\nu)(5-3\nu)}$ and $\nu \in (-1, -\frac{1}{3})$.

- Two **degenerate** kernels:

$$\text{spec}\left(P^{(3)}(-1)\right) = \{1\,, 1\,, -1\}\,, \quad \text{spec}\left(P^{(3)}\,(-1/3)\right) = \left\{\frac{5}{3}\,, -\frac{1}{3}\,, -\frac{1}{3}\right\}.$$

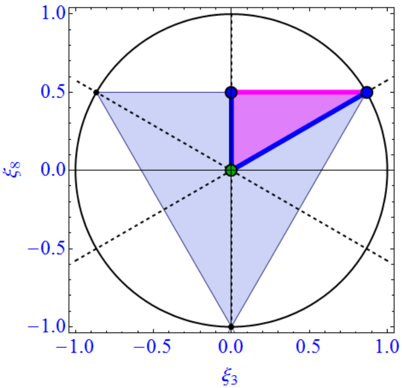
- The spectrum of **singular** kernel:

$$\text{spec}\left(P^{(3)}_{det=0}\right) = \left\{\frac{1+\sqrt{5}}{2}\,, 0\,, \frac{1-\sqrt{5}}{2}\right\}, \quad tr\left([P^{(3)}_{det=0}]^m\right) = \mathcal{L}_m\,,$$

where the m-th **Lucas number** $\mathcal{L}_m = \phi^m + (-\phi)^{-m}$ and $\phi = \frac{1+\sqrt{5}}{2}$.

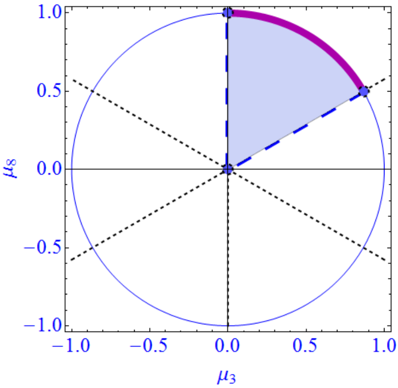
The ordering of the **density matrix** eigenvalues $\boxed{1 \geq r_1 \geq r_2 \geq r_3 \geq 0}$ and condition $\sum r_i = 1$ lead to

$$\boxed{\xi_3 \geq 0\,, \quad \xi_8 \geq \frac{\xi_3}{\sqrt{3}}}\,.$$



The ordering of the **SW kernel** eigenvalues $\boxed{\pi_1 \geq \pi_2 \geq \pi_3}$ and condition $\sum \mu_i^2 = 1$ lead to

$$\boxed{\mu_3 = \sin\zeta\,, \quad \mu_8 = \cos\zeta\,, \quad 0 \leq \zeta \leq \frac{\pi}{3}}\,.$$



Quatrit kernel and its fundamental region

A generic **quatrit** ($N = 4$) state is given by the density matrix

$$\varrho_{quatrit} = \frac{1}{4}\left(I + \sqrt{6}\sum_{\nu=1}^{15} \xi_\nu \lambda_\nu\right) .$$

The **Stratonovich-Weyl kernel**

$$\Delta(\Omega_N|\boldsymbol{\nu}) = U(\Omega_N)\,\frac{1}{4}\left[I + \sqrt{30}\,(\mu_3\lambda_3 + \mu_8\lambda_8 + \mu_{15}\lambda_{15})\right]\,U(\Omega_N)^\dagger .$$

The **Wigner function** of a quatrit

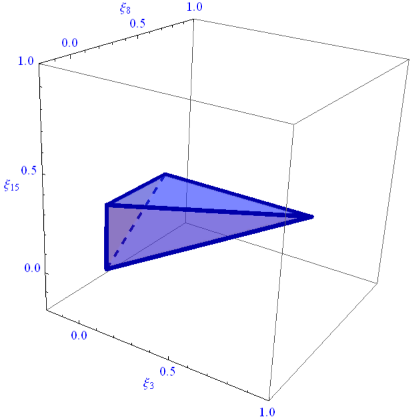
$$W^{(\boldsymbol{\nu})}_{\boldsymbol{\xi}}(\Omega_4) = \frac{1}{4} + \frac{3\sqrt{5}}{4}\left[\mu_3(\boldsymbol{n}^{(3)},\boldsymbol{\xi}) + \mu_8(\boldsymbol{n}^{(8)},\boldsymbol{\xi}) + \mu_{15}(\boldsymbol{n}^{(15)},\boldsymbol{\xi})\right] ,$$

with

$$n_\nu^{(3,8,15)} = \frac{1}{2}\mathrm{tr}\left[U\lambda_{3,8,15}U^\dagger\lambda_\nu\right] .$$

Quatrit density matrix

In a quatrit case, there are 24 ways of the spec $(\varrho_{quatrit}) = \{r_1, r_2, r_3, r_4\}$ ordering.



The fixed order of the eigenvalues

$$1 \geq r_1 \geq r_2 \geq r_3 \geq r_4 \geq 0\,,$$
$$0 \leq r_i \leq 1\,, \qquad \sum r_i = 1\,,$$

leads to

$$0 \leq \xi_3 \leq \sqrt{2/3}\,,$$
$$\frac{\xi_3}{\sqrt{3}} \leq \xi_8 \leq \sqrt{2}/3\,,$$
$$\frac{\xi_8}{\sqrt{2}} \leq \xi_{15} \leq 1/3\,.$$

The **master equations**

$$tr\left(\Delta(\Omega)\right) = 1\,, \qquad tr\left(\Delta(\Omega)^2\right) = 4$$

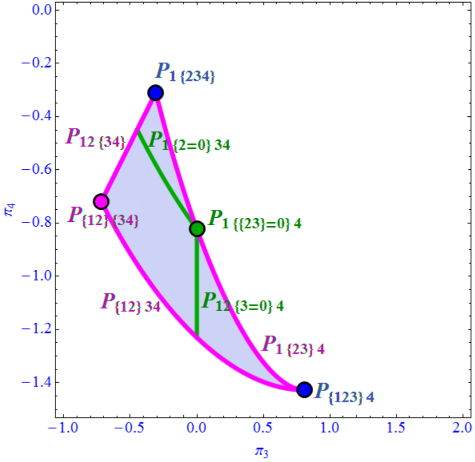
determine two-parametric family of kernels $P^{(4)}$ with $\pi_1 \geq \pi_2 \geq \pi_3 \geq \pi_4$:

- **Generic** kernel:

$$\text{spec}\left(P^{(4)}(\pi_3, \pi_4)\right) = \left\{\frac{\gamma+\delta}{2}\,, \frac{\gamma-\delta}{2}\,, \pi_3\,, \pi_4\right\}\,,$$

where

$$\gamma = 1 - \pi_3 - \pi_4\,, \quad \delta = \sqrt{8 - 2(\pi_3^2 + \pi_4^2) - \gamma^2}\,.$$



Note that

$$\mathcal{R}_\mathbf{m} = \mathcal{R}_{m-1} + \tfrac{3}{2}\mathcal{R}_{m-2}\,, \ \mathcal{R}_1 = 1\,, \mathcal{R}_2 = 4\,;$$
$$\mathcal{L}_\mathbf{m} = \mathcal{L}_{m-1} + \mathcal{L}_{m-2}\,, \quad \mathcal{L}_1 = 2\,, \mathcal{L}_2 = 1\,.$$

- **Degenerate** kernels:
  - Triple degenerate

$$P^{(4)}_{\{123\}4} : \pi_1 = \pi_2 = \pi_3 \neq \pi_4\,,$$
$$P^{(4)}_{1\{234\}} : \pi_1 \neq \pi_2 = \pi_3 = \pi_4\,.$$

  - Double degenerate

$$P_{\{12\}\{34\}} : \pi_1 = \pi_2 \neq \pi_3 = \pi_4\,,$$
$$P_{\{12\}34} : \pi_1 = \pi_2 \neq \pi_3 \neq \pi_4\,,$$
$$P_{1\{23\}4} : \pi_1 \neq \pi_2 = \pi_3 \neq \pi_4\,,$$
$$P_{12\{34\}} : \pi_1 \neq \pi_2 \neq \pi_3 = \pi_4\,.$$

- **Singular** kernels

$$P_{1\{2=0\}34} : \pi_1 \neq \pi_2 = 0 \neq \pi_3 \neq \pi_4\,,$$
$$P_{12\{3=0\}4} : \pi_1 \neq \pi_2 = 0 \neq \pi_3 \neq \pi_4\,,$$
$$P_{1\{\{23\}=0\}4} : \pi_1 \neq \pi_2 \neq \pi_3 = 0 \neq \pi_4,$$

with $tr\left(P^m_{1\{\{23\}=0\}4}\right) = \mathcal{R}_m\,.$

Parameterizing μ by two spherical coordinates

$$\boxed{\mu_3 = \sin\psi_1 \sin\psi_2\,, \quad \mu_8 = \sin\psi_1 \cos\psi_2\,, \quad \mu_{15} = \cos\psi_1}$$

and using the constraints coming from the requirement of a decreasing order of the SW kernel's eigenvalues

$$\mu_3 \geq 0\,, \qquad \mu_8 \geq \frac{\mu_3}{\sqrt{3}}\,, \qquad \mu_{15} \geq \frac{\mu_8}{\sqrt{2}}\,,$$

we have:

$$\left[\begin{array}{l} \begin{cases} \psi_2 \in \left(0, \frac{\pi}{3}\right]\,, \\ 0 < \psi_1 \leq arccot\left(\cos\psi_2/\sqrt{2}\right)\,; \end{cases} \\ \\ \begin{cases} \psi_2 = 0\,, \\ 0 < \psi_1 \leq arccot\left(1/\sqrt{2}\right)\,; \end{cases} \\ \\ \psi_1 = 0\,. \end{array}\right.$$

(See Figure 1)

Girard's theorem: the spherical excess of a triangle determines the solid angle

$$\pi/2 + \pi/3 + \pi/3 - \pi = 4\pi/24\,.$$

Any fixed order of eigenvalues corresponds to one of 24 possible ways to tessellate a sphere.

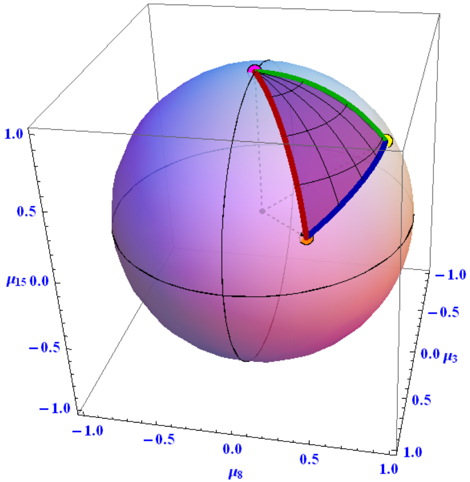


Figure 1: Möbius $(2,3,3)$ triangle with $(\pi/2\,,\pi/3\,,\pi/3)$ angles.

Conclusions

An ambiguity in the master equation's solution for Stratonovich-Weyl kernel is analyzed and the corresponding moduli spaces of the Wigner QPDF is determined for $N = 3, 4$ quantum systems:

- for the qutrit the moduli space is the $\frac{\pi}{3}$ arc of the unit circle,
- for the quatrit the moduli space is $(2, 3, 3)$ Möbius triangle.

The basic goal of our further studies is

understanding of a physical meaning of the Wigner function moduli space.

Thank you for attention