A universality theorem for stressable graphs in the plane

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Universality theorems (in the sense of N. Mnëv) claim that the realization space of a combinatorial object (a point configuration, a hyperplane arrangement, a convex polytope, etc.) can be arbitrarily complicated. In the paper, we prove a universality theorem for a graph in the plane with a collection of signs of its possible equilibrium stresses ("oriented matroid of stresses").

This research is motivated by the Grassmanian stratification (Gelfand, Goresky, MacPherson, Serganova) and a recent series of papers on stratifications of configuration spaces of tensegrities (Doray, Karpenkov, Schepers, Servatius).

Here are details: let $\Gamma = (V, E)$ be a graph without loops and multiple edges, where $V = \{v_1, ..., v_m\}$ is the set of vertices, and E is the set of edges. A *realization* of Γ is a map $p : V \to \mathbb{R}^2$ such that $(ij) \in E$ implies $p(v_i) \neq p(v_j)$. We abbreviate $p(v_i)$ as p_i.

A *stress* $\mathfrak{s}$ on a realization (Γ, p) is an assignment of real scalars $\mathfrak{s}(i, j)$ to the edges. One imagines that each edge is turned to a (either compressed or extended) spring. A stress $\mathfrak{s}$ is called a *self-stress*, or an *equilibrium stress*, if at every vertex p_i, the sum of the forces produced by the springs vanishes:

$$\sum_{(ij)\in E} \mathfrak{s}(i, j)\mathbf{u}_{ij} = 0.$$

Here $\mathbf{u}_{ij} = \frac{p_i - p_j}{|p_i - p_j|}$ is the unit vector pointing from p_j to p_i.

Given realization (Γ, p), the set of all self-stresses $\mathfrak{S}(\Gamma, p)$ is a linear space which naturally embeds in $\mathbb{R}^e$, where $e = |E|$. Set $\mathcal{M}(\Gamma, p) := SIGN(\mathfrak{S}(\Gamma, p))$. In simple words, we do the following: enumerate somehow the edges of the graph, and for each non-trivial stress, list the signs of its values on all the edges. We obtain a collection of strings (elements of $(+, -, 0)^e$).

Given a graph Γ and an oriented matroid $\mathcal{M}$, define the *realization space* of the pair $(\Gamma, \mathcal{M})$ as the space of all realizations of Γ that yield the oriented matroid $\mathcal{M}$. We factorize the space by the action of the general linear group:

$$\mathcal{R}(\Gamma, \mathcal{M}) = \{p : \mathcal{M}(\Gamma, p) = \mathcal{M}\}/GL(2).$$

In general, semialgebraic sets are subsets of some Euclidean space $\mathbb{R}^N$ defined by polynomial equations and inequalities. A semialgebraic set is called a *fat basic primary semialgebraic set (FBP semialgebraic set)* if there are no defining equations, all the defining inequalities are strict, and the coefficients of all the defining polynomials are rational.

Our **main result** is: For each FBP semialgebraic set A, there exists a graph Γ and an oriented matroid $\mathcal{M}$ such that the realization space $\mathcal{R}(\Gamma, \mathcal{M})$ is stably equivalent to A.

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