Computer algebra aided generation of a mimetic difference scheme for 3D steady Stokes flow

Vladimir P. Gerdt, Yury A. Blinkov and Xiaojing Zhang

In paper [1] two of us suggested an algorithmic approach to generation of finite difference schemes for polynomial nonlinear differential equations on regular grids and applied it in [2] to generation of difference schemes for 2D incompressible Navier-Stokes equations. Then, in [3, 4] the novel concept of s(trong)-consistency which strengthens the universally adopted concept of consistency for difference schemes was introduced and in [5, 6, 7] for 2D incompressible Navier-Stokes equations it was shown that s-consistent schemes have better numerical behavior than the s-inconsistent ones. In addition, in [8] and [9] for steady 2D and 3D Stokes flow, respectively, it was demonstrated that the concept of s-consistency plays the key role in construction of the modified equation.

In the present talk we consider algorithmic issues of computer algebra aided generation of a mimetic difference scheme for the governing system of linear partial differential equations for steady Stokes flow whose involutive form is given by

$$\left\{\begin{array}{ll} F^{(1)} := & u_x + v_y + w_z = 0\,, \\ F^{(2)} := & p_x - \frac{1}{\mathrm{Re}}\,(u_{xx} + u_{yy} + u_{zz}) - f^{(1)} = 0\,, \\ F^{(3)} := & p_y - \frac{1}{\mathrm{Re}}\,(v_{xx} + v_{yy} + v_{zz}) - f^{(2)} = 0\,, \\ F^{(4)} := & p_z - \frac{1}{\mathrm{Re}}\,(w_{xx} + w_{yy} + w_{zz}) - f^{(3)} = 0\,, \\ F^{(5)} := & p_{xx} + p_{yy} + p_{zz} - f^{(1)}_x - f^{(2)}_y - f^{(3)}_z = 0\,. \end{array}\right. \tag{1}$$

where x, y, z are the independent variables; the velocities u, v and w, the pressure p, and the external forces $f^{(1)}$,$f^{(2)}$ and $f^{(3)}$ are the dependent variables; the constant Re is the Reynolds number and $\Delta := \partial_{xx} + \partial_{yy} + \partial_{zz}$ is the Laplace operator. The equation $F^{(5)} = 0$ is integrability condition to Eqs. $\{F^{(i)} = 0 \mid 1 \leq i \leq 4\}$ called the pressure Poisson equation.

The system (1) possesses the permutational symmetry

$$\{x, u, f^{(1)}\} \longleftrightarrow \{y, v, f^{(2)}\} \longleftrightarrow \{z, w, f^{(3)}\}\,. \tag{2}$$

The work is supported in part by the Russian Foundation for Basic Research (grant No. 18-51-18005).

To preserve this symmetry at the discrete level we consider the Cartesian grid with the spacing h and apply the method of paper [1] to construct a difference scheme for (1) which is symmetric under permutation (2), strongly consistent and conservative. The s-consistency means inheritance by the scheme such important algebraic property of (1) as vanishing of any element in the differential ideal generated by the polynomials in (1) on any common solution to these equations. This means that for the difference scheme any element of the difference ideal generated by the polynomials in the scheme approximates a polynomial in the differential ideal. Besides, we want to have the scheme to be conservative, i.e. having the conservation law properties inherent in (1).

The procedure of the scheme generation described in paper [9] yields the following results

$$
\begin{cases}
\tilde{F}^{(1)} := & \dfrac{u_{j+2,\,k+1,\,l+1} - u_{j,\,k+1,\,l+1}}{2h} + \dfrac{v_{j+1,\,k+2,\,l+1} - v_{j+1,\,k,\,l+1}}{2h} \\
& + \dfrac{w_{j+1,\,k+1,\,l+2} - v_{j+1,\,k+1,\,l}}{2h} = 0\,, \\
\tilde{F}^{(2)} := & \dfrac{p_{j+2,\,k+1,\,l+1} - p_{j,\,k+1,\,l+1}}{2h} - \dfrac{1}{\mathrm{Re}}\Delta_1\left(u_{j,k,l}\right) - f^{(1)}_{j+1,\,k+1,\,l+1} = 0\,, \\
\tilde{F}^{(3)} := & \dfrac{p_{j+1,\,k+2,\,l+1} - p_{j+1,\,k,\,l+1}}{2h} - \dfrac{1}{\mathrm{Re}}\Delta_1\left(v_{j,k,l}\right) - f^{(2)}_{j+1,\,k+1,\,l+1} = 0\,, \\
\tilde{F}^{(4)} := & \dfrac{p_{j+1,\,k+1,\,l+2} - p_{j+1,\,k+1,\,l}}{2h} - \dfrac{1}{\mathrm{Re}}\Delta_1\left(w_{j,k,l}\right) - f^{(3)}_{j+1,\,k+1,\,l+1} = 0\,, \\
\tilde{F}^{(5)} := & -\dfrac{f^{(1)}_{j+3,\,k+2,\,l+2} - f^{(1)}_{j+1,\,k+2,\,l+2}}{2h} - \dfrac{f^{(2)}_{j+2,\,k+3,\,l+2} - f^{(2)}_{j+2,\,k+1,\,l+2}}{2h} \\
& -\dfrac{f^{(3)}_{j+2,\,k+2,\,l+3} - f^{(3)}_{j+2,\,k+2,\,l+1}}{2h} + \Delta_2\left(p_{j,k,l}\right) = 0\,,
\end{cases}
$$

where where Δ_1 and Δ_2 are finite difference discretizations of the Laplace operator acting on a grid function $g_{j,\,k,\,l}$ as

$$
\begin{aligned}
\Delta_1\left(g_{j,\,k,\,l}\right) := & \quad \frac{g_{j+2,\,k+1,\,l+1} + g_{j+1,\,k+2,\,l+1} + g_{j+1,\,k+1,\,l+2} - 6g_{j+1,\,k+1,\,l+1}}{h^2} \\
& + \frac{g_{j+1,\,k,\,l+1} + g_{j,\,k+1,\,l+1} + g_{j+1,\,k+1,\,l}}{h^2}\,, \\
\Delta_2\left(g_{j,\,k,\,l}\right) := & \quad \frac{g_{j+4,\,k+2,\,l+2} + g_{j+2,\,k+4,\,l+2} + g_{j+2,\,k+2,\,l+4} - 6g_{j+2,\,k+2,\,l+2}}{4h^2} \\
& + \frac{g_{j+2,\,k,\,l+2} + g_{j,\,k+2,\,l+2} + g_{j+2,\,k+2,\,l}}{4h^2}\,.
\end{aligned}
$$

Note, that the replaces Δ_2 with Δ_1 preserves consistency of the scheme but violate its s-consistency. Based on the s-consistent scheme one can compute its modified equation [9] which allows to analyze the order of accuracy of the scheme. The obtained scheme is of the second order.

References

[1] V. P. Gerdt, Yu. A. Blinkov, and V. V. Mozzhilkin. Gröbner Bases and Generation of Difference Schemes for Partial Differential Equations. *Symmetry, Integrability and Geometry: Methods and Applications*, 2:26, 2006. arXiv:math.RA/0605334

[2] V. P. Gerdt and Yu. A. Blinkov. Involution and difference schemes for the Navier-Stokes equations. In *Computer Algebra in Scientific Computing*, volume 5743 of *Lecture Notes in Computer Science*, pages 94–105. Springer Berlin / Heidelberg, 2009.

[3] V. P. Gerdt and D. Robertz. Consistency of Finite Difference Approximations for Linear PDE Systems and its Algorithmic Verification. In: S. Watt (ed.). *Proceedings of ISSAC 2010*, pp. 53–59. Association for Computing Machinery, 2010.

[4] V. P. Gerdt. *Consistency Analysis of Finite Difference Approximations to PDE Systems.* In *Mathematical Modelling in Computational Physics*, volume 7125 of *Lecture Notes in Computer Science*, pages 28–42. Springer, Berlin, 2012. arXiv:math.AP/1107.4269

[5] P. Amodio, Yu. A. Blinkov, V. P. Gerdt, and R. La Scala. On Consistency of Finite Difference Approximations to the Navier-Stokes Equations. In *Computer Algebra in Scientific Computing*, volume 8136 of *Lecture Notes in Computer Science*, pages 46–60. Springer Berlin / Heidelberg, 2013. arXiv:math.NA/1307.0914

[6] P. Amodio, Yu. A. Blinkov, V. P. Gerdt and R. La Scala. *Algebraic construction and numerical behavior of a new s-consistent difference scheme for the 2D Navier–Stokes equations.* Applied Mathematics and Computation, 314, 408–421, 2017.

[7] D. L. Michels,V. P. Gerdt,Yu. A. Blinkov and D.A.Lyakhov. *On the consistency analysis of finite difference approximations.* Zapiski Nauchnyh Seminarov POMI, 468, 249–266, 2018.

[8] Yu. A. Blinkov, V. P. Gerdt, D. A. Lyakhov and D. L. Michels. *A Strongly Consistent Finite Difference Scheme for Steady Stokes Flow and its Modified Equations.* In *Computer Algebra in Scientific Computing*, volume 11077 of *Lecture Notes in Computer Science*, , pp.67–81, Springer, Cham, 2018.

[9] X. Zhang, V. P. Gerdt and Yu. A. Blinkov. *Algebraic Construction of a Strongly Consistent, Permutationally Symmetric and Conservative Difference Scheme for 3D Steady Stokes Flow.* Symmetry, 11, 269, (15 pages), 2019. https://www.mdpi.com/2073-8994/11/2/269

Vladimir P. Gerdt
Joint Institute for Nuclear Research
Dubna, Russia
e-mail: gerdt@jinr.ru

Yury A. Blinkov
Saratov State University
Saratov, Russia
e-mail: blinkovua@info.sgu.ru

Xiaojing Zhang
University of Chinese Academy of Sciences
Beijing, China
e-mail: zhangxiaojing17@mails.ucas.ac.cn