Computer algebra aided generation of a mimetic
difference scheme for 3D steady Stokes flow

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In paper [1] two of us suggested an algorithmic approach to generation of
finite difference schemes for polynomial nonlinear differential equations on regular
grids and applied it in [2] to generation of difference schemes for 2D incompressible
Navier-Stokes equations. Then, in [3, 4] the novel concept of s(trong)-consistency
which strengthens the universally adopted concept of consistency for difference
schemes was introduced and in [5, 6, 7] for 2D incompressible Navier-Stokes equa-
tions it was shown that s-consistent schemes have better numerical behavior than
the s-inconsistent ones. In addition, in [8] and [9] for steady 2D and 3D Stokes
flow, respectively, it was demonstrated that the concept of s-consistency plays the
key role in construction of the modified equation.

In the present talk we consider algorithmic issues of computer algebra aided
generation of a mimetic difference scheme for the governing system of linear partial
differential equations for steady Stokes flow whose involutive form is given by

\[
\begin{align*}
F^{(1)} &:= u_x + v_y + w_z = 0, \\
F^{(2)} &:= p_x - \frac{1}{\text{Re}} (u_{xx} + u_{yy} + u_{zz}) - f^{(1)} = 0, \\
F^{(3)} &:= p_y - \frac{1}{\text{Re}} (v_{xx} + v_{yy} + v_{zz}) - f^{(2)} = 0, \\
F^{(4)} &:= p_z - \frac{1}{\text{Re}} (w_{xx} + w_{yy} + w_{zz}) - f^{(3)} = 0, \\
F^{(5)} &:= p_{xx} + p_{yy} + p_{zz} - f_x^{(1)} - f_y^{(2)} - f_z^{(3)} = 0.
\end{align*}
\]

where \( x, y, z \) are the independent variables; the velocities \( u, v \) and \( w \), the pressure \( p \),
and the external forces \( f^{(1)}, f^{(2)} \) and \( f^{(3)} \) are the dependent variables; the constant
\( \text{Re} \) is the Reynolds number and \( \Delta := \partial_{xx} + \partial_{yy} + \partial_{zz} \) is the Laplace operator.
The equation \( F^{(5)} = 0 \) is integrability condition to Eqs. \( \{ F^{(i)} = 0 \mid 1 \leq i \leq 4 \} \) called
the pressure Poisson equation.

The system (1) possesses the permutational symmetry

\[
\{ x, u, f^{(1)} \} \longleftrightarrow \{ y, v, f^{(2)} \} \longleftrightarrow \{ z, w, f^{(3)} \}.
\]

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18005).
To preserve this symmetry at the discrete level we consider the Cartesian grid with the spacing \( h \) and apply the method of paper [1] to construct a difference scheme for (1) which is symmetric under permutation (2), strongly consistent and conservative. The s-consistency means inheritance by the scheme such important algebraic property of (1) as vanishing of any element in the differential ideal generated by the polynomials in (1) on any common solution to these equations. This means that for the difference scheme any element of the difference ideal generated by the polynomials in the scheme approximates a polynomial in the differential ideal. Besides, we want to have the scheme to be conservative, i.e. having the conservation law properties inherent in (1).

The procedure of the scheme generation described in paper [9] yields the following results

\[
\begin{align*}
\tilde{F}^{(1)} &:= \frac{u_{j+2,k+1,l+1} - u_{j,k+1,l+1}}{2h} + \frac{v_{j+1,k+2,l+1} - v_{j+1,k,l+1}}{2h} = 0, \\
\tilde{F}^{(2)} &:= \frac{p_{j+2,k+1,l+1} - p_{j,k+1,l+1}}{2h} - \frac{1}{\Re} \Delta_1(u_{j,k,l}) - f^{(1)}_{j+1,k+1,l+1} = 0, \\
\tilde{F}^{(3)} &:= \frac{p_{j+1,k+2,l+1} - p_{j+1,k,l+1}}{2h} - \frac{1}{\Re} \Delta_1(v_{j,k,l}) - f^{(2)}_{j+1,k+1,l+1} = 0, \\
\tilde{F}^{(4)} &:= \frac{p_{j+1,k+1,l+2} - p_{j+1,k+1,l+1}}{2h} - \frac{1}{\Re} \Delta_1(w_{j,k,l}) - f^{(3)}_{j+1,k+1,l+1} = 0, \\
\tilde{F}^{(5)} &:= -\frac{f^{(1)}_{j+3,k+2,l+2} - f^{(1)}_{j+1,k+2,l+2}}{2h} - \frac{f^{(2)}_{j+2,k+3,l+2} - f^{(2)}_{j+2,k+1,l+2}}{2h} + \Delta_2(p_{j,k,l}) = 0,
\end{align*}
\]

where where \( \Delta_1 \) and \( \Delta_2 \) are finite difference discretizations of the Laplace operator acting on a grid function \( g_{j,k,l} \) as

\[
\begin{align*}
\Delta_1 \left( g_{j,k,l} \right) &:= \frac{g_{j+2,k+1,l+1} + g_{j+1,k+2,l+1} + g_{j+1,k+1,l+2} - 6g_{j+1,k+1,l+1}}{h^2} + \frac{g_{j+1,k,l+1} + g_{j,k+1,l+1} + g_{j+1,k+1,l}}{h^2}, \\
\Delta_2 \left( g_{j,k,l} \right) &:= \frac{g_{j+4,k+2,l+2} + g_{j+2,k+4,l+2} + g_{j+2,k+2,l+4} - 6g_{j+2,k+2,l+2}}{4h^2} + \frac{g_{j+2,k,l+2} + g_{j,k+2,l+2} + g_{j+2,k+2,l}}{4h^2}.
\end{align*}
\]

Note, that the replaces \( \Delta_2 \) with \( \Delta_1 \) preserves consistency of the scheme but violate its s-consistency. Based on the s-consistent scheme one can compute its modified equation [9] which allows to analyze the order of accuracy of the scheme. The obtained scheme is of the second order.
References


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