A Numerical Roadmap Algorithm for Smooth Bounded Real Algebraic Surface

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Abstract. For a smooth bounded real algebraic surface in three-dimensional space, a roadmap of it is a one-dimensional semi-algebraic subset of the surface whose intersection with each connected component of the surface is nonempty and semi-algebraically connected. In this paper, we introduce the notion of a numerical roadmap of a surface, which is a set of polygonal chains such that there is a bijective map between the chains and the connected components of a given roadmap of the surface. Moreover, the chains are ϵ-close to the connected components. We present an algorithm to compute such a numerical roadmap through constructing a topological graph. The topological graph also enables us to compute a more intrinsic connectivity graph of the roadmap, which is important for applications such as finding a connected path between two points on the surface, as well as grouping witness points of the surface into different connected components.

Introduction

Roadmap was introduced by Canny [6] in 1987 for solving robot motion planning problems. Since then, the initial roadmap algorithm has been improved by himself and many others [17, 14, 11, 20]. For a polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree d, Basu et al. [3] recently proposed a symbolic roadmap algorithm with a complexity of $d^{O(n\sqrt{n})}$ based on the earlier work of Safey el Din and Schost [10].

The problem of deciding whether two points belong to the same connected component of a semi-algebraic set is a fundamental problem in real algebraic geometry. Such problems can be solved by computing cylindrical algebraic decompositions (CAD) [9, 21], Today, the implementations of CAD are widely available thanks to different softwares QEPCAD, Mathematica, REDLOG, SyNRAC, `RegularChains`. The complexity for computing CAD is double exponential in the number of variables. In contrast, the roadmap based algorithms have a single exponential complexity and thus provides a theoretically more powerful tool for solving the connectivity problems in semi-algebraic geometry, such as determining if two points in a semi-algebraic set are connected, or counting the connected components of a semi-algebraic set. However, to the best of our knowledge, there have been no exact implementations of the roadmap algorithms.

Today, many problems in computational real algebraic geometry can be addressed in a different way by means of powerful tools from numerical algebraic geometry [22, 13], such as computing witness points for connected components of real varieties [12, 24, 23]. Despite some attempts [15, 16] for numerically computing roadmaps, the notion of numerical roadmap has not been rigorously defined until now. And there are still no complete numerical implementations of the roadmap algorithms.

It is natural to develop a numerical roadmap algorithm since the roadmap as a one-dimensional semi-algebraic set can be approximated by polygonal chains based on numerical continuation techniques. The difficulty for developing such an algorithm would be to guarantee there is a one-to-one correspondence between the connected components of the roadmap and the connected components of its polygonal chains approximation. Indeed, to achieve this, from a numerical point of view, one has to overcome some obstacles, such as avoiding curve jumping [5, 4, 19, 25, 24, 7] and handling singularities [1, 18, 8].

In this work, for a given smooth bounded real algebraic surface, we provide a numerical version of the classical roadmap algorithm introduced by Canny [6, 2]. We introduce the concept of a numerical roadmap and propose an algorithm to generate it through constructing some graphs joining the silhouette of the surface and slice curves passing through the critical points of a projection map on the silhouette. The slice curves may have singularities, which is handled based on a technique for tracing singular planar curves [7].

1. Main results

Throughout this paper, let $f \in \mathbb{R}[X_1, X_2, X_3]$ and $\mathrm{Z}_{\mathbb{R}}(f)$ (or simply Z if no confusion arises) be its zero set in $\mathbb{R}^3$. We assume:

(A_1) $\mathrm{Z}_{\mathbb{R}}(f)$ is nonempty[1] and bounded.

(A_2) f attains full rank at any point of $\mathrm{Z}_{\mathbb{R}}(f)$.

Thus, $\mathrm{Z}_{\mathbb{R}}(f)$ is a smooth bounded surface in $\mathbb{R}^3$. In addition, without lost of generality, we enforce the assumption (A_3): the critical set of π_{12} is a manifold, holds.

Definition 1. *A one-dimensional semi-algebraic subset RM of* Z *is called a* **roadmap** *of* Z *if the following two properties are satisfied:*

(R_1) *The intersection of* Z *with each semi-algebraically connected component of RM is nonempty and semi-algebraically connected.*

(R_2) *For every $c \in \mathbb{R}$, every semi-algebraically connected component of Z_c has nonempty intersection with RM.*

[1] If $\mathrm{Z}_{\mathbb{R}}(f)$ is empty, its roadmap will be empty. We make this assumption for simplicity.

Definition 2. *The critical set* $\Sigma(\pi_{12}|\mathrm{Z})$ *is called the* **silhouette** *of* Z. *For any* $c \in \mathbb{R}$, *if* $\mathrm{Z}_c \neq \emptyset$ *and* $\dim(\mathrm{Z}_c) \leq 1$, *it is called a* **slice curve** *of* Z.

Let SI be the silhouette of Z. By Sard's Theorem, there are only finitely many critical values of $X_1 : SI \to \mathbb{R}$. Let $c_1 < \cdots < c_r$ be all the X_1-critical values of SI, where c_1 and c_r are respectively the minimal and maximal value of X_1 on Z. Let $SL_0 := \cup_{i=2}^{r-1} \mathrm{Z}_{c_i}$ and $RM_0 := SI \cup SL_0$.

We call each hyperplane $X_1 = c_i$, $i = 2, \ldots, r-1$ a **distinguished hyperplane** and each point in $\cup_{i=2}^{r-1} \mathrm{Z}_{c_i} \cap SI$ a **distinguished point**.

Theorem 1 ([6, 2]). *The set* RM_0 *is a roadmap of* Z.

Definition 3. *Given a roadmap* RM *of* $\mathrm{Z}_{\mathbb{R}}(f)$ *and a given precision* $\epsilon \in R$. *A set* $\mathcal{S}$ *of polygonal chains is called a* **numerical roadmap** *of* $Z_{\mathbb{R}}(f)$ *(*ϵ*-close to* RM*) if there is a bijection map* $\mathfrak{m}$ *between* $\mathcal{S}$ *and the connected components of* RM *such that the Hausdorff distance* $d_H(P, \mathfrak{m}(P)) \leq \epsilon$ *holds for each* $P \in \mathcal{S}$.

Note that if we make a small ϵ perturbation to a surface Z, the number of connected components of Z may change if the Hausdorff distance between two components is less than ϵ. Thus, in the rest of this paper, we would assume that the Hausdorff distance between any two connected components of a surface is much larger than ϵ. Moreover, we assume that the distance between any X_1-critical points of the silhouette is much larger than ϵ. We name the two numerical assumptions as (A_4).

The main idea is to use the distinguished points as well as the critical points on the hyperplane $X_1 = c_1$ and $X_1 = c_r$, which are the X_1-global extremum points on the surface, as seed points to generate some neighbor points. These neighbor points will be used as initial points for curve tracing. The distinguished points, critical points on the hyperplane $X_1 = c_1$ and $X_1 = c_r$ and neighbor points will be the vertices of the graph and they are connected by edges representing the curve segments between them. We call this graph a **topological graph** of the roadmap RM_0. We can refine this graph to obtain an **approximate graph** of RM_0 and build the **connectivity graph** of RM_0.

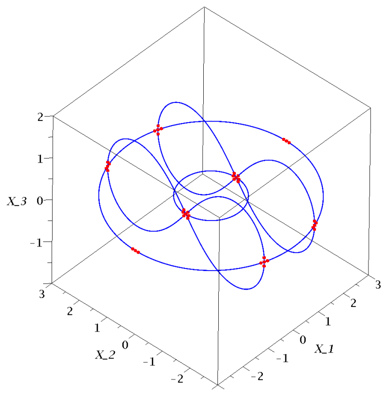
Theorem 2. *Under Assumptions* $(A_1), \ldots, (A_4)$, *one can control errors of staring points and prediction-correction in curve tracing to compute an* **approximate graph** *of* RM_0, *whose zero set is a numerical roadmap* ϵ*-close to the roadmap* RM_0 *of* Z.

2. Examples

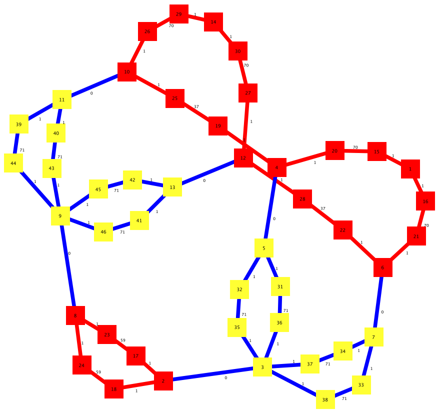
Example 1. *Consider the torus surface defined by*

$$f := \left(x^2 + y^2 + z^2 + 3\right)^2 - 16\,x^2 - 16\,y^2.$$

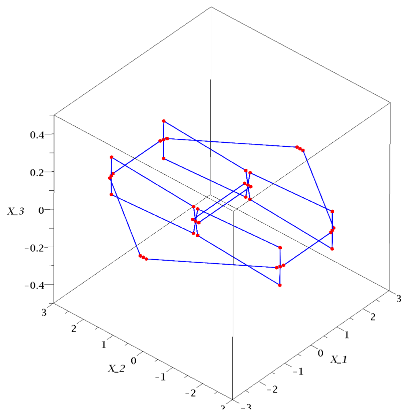
Example 2. *The Chub's surface defined by* $f := x^4 + y^4 + z^4 - x^2 - y^2 - z^2 + 0.5$ *is a singular surface. We replaces the constant coefficient of* f *by* 0.4 *(resp.* 0.6*) and the perturbed polynomials as* $Chub_1$ *(resp.* $Chub_2$*).*



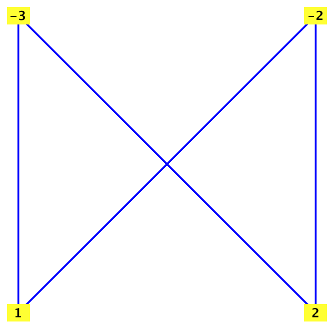
A numerical roadmap of the torus.



The computed topological graph of the torus.



Zero set of the topological graph of the torus.



The connectivity graph of the torus.

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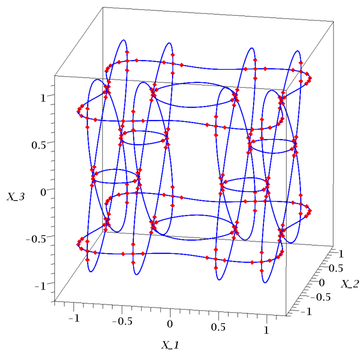
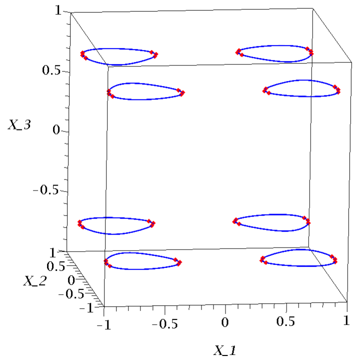


A numerical roadmap of $Chub_1$.



A numerical roadmap of $Chub_2$.

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