Sketch on quaternionic Lorentz transformations

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Abstract. Lorentz transformations are decomposed into a linear combination of two orthogonal transformations. In this way a two-term expression of Lorentz transformations by means of quaternions is proposed. An analytical solution to the problem of finding eigenvectors is given. The conditions for the existence of eigenvectors are specified. A quartet of eigenvectors that occurs when rotational axis is orthogonal to velocity direction is obtained. The accompanying relativistic velocity addition is discussed.

Introduction

W.R. Hamilton had discovered quaternions in the 19 century in order best to describe real four-dimensional spacetime $\mathbb{R}^4$, supplied with a *cross product* $[\nu, n]$ of spatial vectors ν and n. The main advantage of quaternions is that they allow working with linear transformations of 4-dimensional Euclidian space without explicitly introducing a standard orthonormal basis and matrix representation of a linear operator. The use of quaternion multiplication provides concise calculations. So, the rotation V of a 3-dimensional space is elegantly described through multiplication of quaternions as $V\{u\} = bu\bar{b}$, where: $b = i_0 \cos\frac{\varphi}{2} + \nu \sin\frac{\varphi}{2}$; $\bar{b} = i_0 \cos\frac{\varphi}{2} - \nu \sin\frac{\varphi}{2}$; i_0 is the multiplicative unity; ν is the *unit* vector of unit length along rotational axis, such that $\sqrt{(\nu,\nu)} = 1$ and φ is the rotational angle [1].

In the quaternion space, the Lorentz transformations are expressed only slightly more complicated than the rotation V.

1. Lorentz transformations in terms of quaternions

The Lorentz transformations $\mathcal{L}$ are defined as a linear transformation of the space of quaternions u, v that preserves the real inner product $(u, \bar{v})$ of one *conjugated* vector $\bar{v} = 2(v, i_0) - v$ by another vector u: $(\mathcal{L}\{u\}, \overline{\mathcal{L}\{v\}}) = (u, \bar{v})$. The Lorentz transformations $\mathcal{L}$ is decomposed into the two simple transformations V and L as in [2], so that $\mathcal{L}\{u\} = \pm VL\{u\}$ or $\mathcal{L}\{\bar{u}\} = \pm VL\{\bar{u}\}$. For brevity, only one option

$\mathcal{L}\{u\} = VL\{u\}$ is treated, where L is the Lorentz boost. It is *screw-symmetrical*: $(L\{u\}, v) = (L\{v\}, u)$ for any u, v.

$L\{u\}$ is expressed in quaternions as:

$$L\{u\} = \bar{a}ua - n\bar{u}\sinh\theta \equiv au\bar{a} - \bar{u}n\sinh\theta, \tag{1}$$

where $a = i_0\cosh\frac{\theta}{2} + n\sinh\frac{\theta}{2}$ and θ is the *rapidity*, such that the velocity vector $\mathbf{v}$ divided by scalar speed of light $\mathbf{c}$ is expressed as $\mathbf{v}/\mathbf{c} = n\tanh\theta$.

It is noteworthy that in [3, 4] the Lorentz boost is described by the truncated formula (1). But this is achieved only due to extra dimensionality, which complicates interpretation.

The dual expression (1) for L and simple quaternion multiplication rules [1, 5] provide easy operation with the Lorentz transformations $\mathcal{L} = VL$ in a coordinate-free way. It is a good exercise to obtain eigenvectors c_k for the transformation $\mathcal{L} = VL$: $\mathcal{L}\{c_k\} = \xi_k c_k$, where ξ_k are real eigenvalues and eigenvector sequence number k starts from 0 and does not exceed 3.

2. Eigenvectors in the general case

It is trivial that the eigenvectors of the transformation $\mathcal{L} = VL$, corresponding to eigenvalues other than 1, are *pseudo-orthogonal* to themselves and to the *invariant* eigenvectors that correspond to $\xi = 1$, e.g. for $\xi_0 \neq 1, \xi_1 \neq 1, \xi_3 = \xi_4 = 1$ the formulae $(c_0, \bar{c}_0) = (c_1, \bar{c}_1) = (c_0, \bar{c}_2) = (c_0, \bar{c}_3) = (c_1, \bar{c}_2) = (c_1, \bar{c}_3) = 0$ are valid. A concomitant fact is that the eigenvalues are pairwise mutually inverse due to the invariance of the equation for ξ with respect to the replacement of ξ by ξ^{-1}.

For easy finding of real eigenvalues, it is convenient to present the general equation for ξ as follows:

$$(\xi - \xi_0)(\xi - \xi_0^{-1})[\xi^2 + \xi(2x - \alpha - \beta) + 1] = 0, \tag{2}$$

where x, which must be outside the interval $(0, 1)$, is found from the equation:

$$x^2 - x\frac{\alpha + \beta}{2} + \frac{\alpha - \beta}{2} - 1 = 0, \tag{3}$$

ξ_0 is found from the equation

$$\frac{\xi_0 + \xi_0^{-1}}{2} = x \tag{4}$$

and $\alpha = (\cosh\theta + 1)(1 + \cos\varphi) \geq 0, \beta = (\nu, n)^2(\cosh\theta - 1)(1 - \cos\varphi) \geq 0$. Concerning the latters it should be noted that in the expressions for α and β, the values of θ and φ are assumed to be non-trivial and *variative*, i.e. both are not fixed for given rotational axis ν and velocity direction n.

With positive α and β the equation (3) for x has at least one required solution $x > 1$. In this case, a pair of mutually inverse eigenvalues other than 1 is available. For each eigenvalue $\xi \neq 1$ the corresponding eigenvector is represented as $i_0 - d$, where $(d, i_0) = 0$, $(d, d) = 1$. In turn, the components of the vector d are calculated by the formulae:

$$(d, \nu) = (\nu, n)\frac{\cosh\theta - 1}{\sinh\theta}\frac{\xi + 1}{\xi - 1}, \; (d, n) = \frac{\xi - \cosh\theta}{\sinh\theta},$$
$$(d, [\nu, n]) = [(d, n) - (\nu, n)(d, \nu)]\frac{\xi \sin\varphi}{\xi\cos\varphi - 1}. \tag{5}$$

3. The case of velocity, orthogonal to the rotational axis

All four eigenvectors are available in the special case of $(\nu, n) = \beta = 0$. In this case, the equation for x becomes trivial:

$$(x - 1)\left(x + 1 - \frac{\alpha}{2}\right) = 0. \tag{6}$$

In the case of $x = \frac{\alpha}{2} - 1$ the equation for finding ξ_0 and $\xi_1 = \xi_0^{-1}$ is expressed by the formula:

$$\xi^2 - \xi(\alpha - 2) + 1 = 0. \tag{7}$$

As follows from the last formula (7), in order to successfully find the target values of ξ_0 and $\xi_1 = \xi_0^{-1}$, the next condition must be satisfied:

$$\left|\sin\frac{\varphi}{2}\right| \le \left|\tanh\frac{\theta}{2}\right| \Leftrightarrow \left|\cos\frac{\varphi}{2}\right|\cosh\frac{\theta}{2} \ge 1 \Leftrightarrow \left|\tan\frac{\varphi}{2}\right| \le \left|\sinh\frac{\theta}{2}\right|, \tag{8}$$

where pair vertical lines denotes absolute value. In the case of $x = 1$, we get the trivial equation for ξ: $(\xi - 1)^2 = 0$ and obtain a pair $\xi_3 = \xi_4 = 1$ of unit values of ξ corresponding to a pair of invariant eigenvectors.

Explicit expressions for eigenvectors and eigenvalues are listed in the table 1, wherein ξ_0 is the solution of (7) under the condition (8).

Notation	Eigenvector	Eigenvalue
c_0	$i_0 - \left(n + [\nu, n]\frac{\xi_0 \sin\varphi}{\xi_0\cos\varphi - 1}\right)\frac{\xi_0 - \cosh\theta}{\sinh\theta}$	ξ_0
c_1	$i_0 - \left(n + [\nu, n]\frac{\sin\varphi}{\cos\varphi - \xi_0}\right)\frac{1 - \xi_0\cosh\theta}{\xi_0\sinh\theta}$	ξ_0^{-1}
c_2	$i_0 + \left(n - [\nu, n]\cot\frac{\varphi}{2}\right)\tanh\frac{\theta}{2}$	1
c_3	ν	1

Table 1. Eigenvector quartet in the case of $(\nu, n) = 0$

Any vector u is trivially decomposed into a linear combination of the listed eigenvectors:

$$u = \frac{c_0(u, \bar{c}_1) + c_1(u, \bar{c}_0)}{(c_0, \bar{c}_1)} + c_2\frac{(u, \bar{c}_2)}{(c_2, \bar{c}_2)} + c_3\frac{(u, \bar{c}_3)}{(c_3, \bar{c}_3)}. \tag{9}$$

Note that the expansion (9) is available only if the condition (8) is fulfilled.

Conclusion

Case $(\nu, n) = 0$ is the most important because it is this case that arises in relativistic addition of velocities, interpreted in terms of Lobachevsky theory [6, 7, 8]. However, according to the authoritative opinion of John Frederick Barrett, "The hyperbolic theory is not at all new and was described by V. Varicak shortly after Einstein's initial work. But it has been ignored now for over 100 years by the mainstream theory." Perhaps, the task of obtaining of the eigenvectors for the Lorentz transformations represented in quaternions, and also in octonions, will be useful for further development in this direction.

In the following papers it will be shown that the decomposition (9) of any vector into eigenvectors is not available for relativistic addition of velocities. Perhaps, professional physicists will give a plausible interpretation for this. In any case, the quaternion technique of working with spatial transformations seems useful for solving modern engineering problems.

References

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