Some properties of doubly symmetric periodic solutions to Hamiltonian system

Alexander Batkhin

Abstract. We consider the structure of doubly symmetric periodic solutions to a Hamiltonian system with two degrees of freedom, which canonical equations of motion are invariant under the action of a forth order discrete group of linear automorphisms of the extended phase space. Structure and bifurcations of doubly symmetric periodic solutions are investigated. It is shown that in the case of period doubling bifurcation there always exists a pair of singly symmetric solutions with double period. Some examples of families of doubly symmetric periodic solutions of the Hill problem and of the restricted three-body problem (in the case of equal masses) are considered.

Introduction

Consider a time-independent non-integrable Hamiltonian system with two degrees of freedom, which canonical equations of motion posses the only first integral $H(\mathbf{z}) = h$, where $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathbb{M} \equiv R^4$ and $H(\mathbf{z})$ is a smooth Hamiltonian function of a system. Let the equations of motion

$$\dot{\mathbf{z}} = J \operatorname{grad} H(\mathbf{z}), \text{ here } J - \text{symplectic unit}, \tag{1}$$

are invariant under the discrete group $\mathcal{G} \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$ of linear automorphisms of the extended phase space $\mathbb{R} \times \mathbb{M}$. Two generators g_1, g_2 of the group $\mathcal{G}$ are involutive operators, i.e. $g_i^2 = \mathrm{id}$, the third non-trivial transformation $g_3 = g_1 \circ g_2 = g_2 \circ g_1$. Then each solution $\mathbf{z}(t, \mathbf{z}_0)$ to equation (1) with initial condition $\mathbf{z}(0) = \mathbf{z}_0$ belongs to one of the following group depending on the type of symmetry.

- **Non-symmetric** solutions, which change under any automorphism g_i,$i = 1, 2, 3$.
- **Singly symmetric** solutions, which are invariant under only one automorphism g_i,$i = 1, 2, 3$.
- **Doubly symmetric** solutions, which are invariant under any automorphism g_i, $i = 1, 2, 3$.

Let $\mathbf{z}(t, \mathbf{z}_0)$ be a periodic solution with period T. In the case of time-independent system solutions $\mathbf{z}(t, \mathbf{z}_0)$ belongs to a family of one-parametric periodic solutions. The parameter of such family is a value of the first integral $H(\mathbf{z})$ of system (1). The family of periodic solutions can be either closed or it has a natural termination. Such characteristics of a periodic solution as its dimension, period T, corresponding value h of the first integral, stability S change smoothly along the family, whereas the type of the symmetry is the global invariant of the family. A family can intersect another one sharing a common solution, but any family can be continued further in an unique way.

Let consider that the phase coordinates $\mathbf{z}$ are chosen in a such manner that the involutive transformation g_i has the form $g_i : (t, \mathbf{z}) \to (\sigma t, G_i \mathbf{z})$, where $\sigma = \pm 1$, G_i is a constant matrix. Let $\Sigma_i \equiv \{\mathbf{z} | g_i(\mathbf{z}) = \mathbf{z}\}$ is an invariant set of the transformation g_i. Then there exist g_i-invariant periodic solutions, which are completely defined by a part of their phase trajectories contained between points $\mathbf{z}_0$ and $\mathbf{z}(T/2)$ lying on the Σ_i.

We study in linear approximation the dynamics near doubly symmetric periodic solution $\mathbf{z}(t, \mathbf{z}_0)$ and provide its bifurcation analysis as well.

1. Properties of doubly symmetric solution

Dynamics of the system (1) in the vicinity of a solution $\mathbf{z}(t)$ is described in linear approximation by matrix $Z(t, \mathbf{z})$, which is the solution to the Cauchy problem of the Poincare variational equation

$$\dot{Z} = J \operatorname{Hess} H(\mathbf{z}) Z, \quad Z_0 = E^4,$$

where $\operatorname{Hess} H(\mathbf{z})$ is the Hessian of function H computed along the solution $\mathbf{z}(t, \mathbf{z}_0)$. Matrix $Z(t)$ is symplectic: $Z^T J Z = J$ and its characteristic polynomial $P(\lambda)$ is reciprocal. Monodromy matrix M of periodic solution $\mathbf{z}(t, \mathbf{z}_0)$ is $Z(T, Z_0)$ with a property $Z(t+T) = Z(t)M$. Eigenvalues ρ_k of matrix M are called *multiplicators*. Monodromy matrix M has the following properties:

- multiplicators ρ_k are mutually complex conjugate and mutually inverse;
- matrix M has eigenvector $\mathbf{v}_1 \equiv J \operatorname{grad} H(\mathbf{z}_0)$, corresponding to multiplicator $\rho_{1,2} = 1$ with multiplicity 2;
- characteristic polynomial $P(\lambda)$ of matrix M is factorized

  $$P(\lambda) = (\lambda - 1)^2 \left(\lambda^2 - 2S\lambda + 1\right),$$

  where S is *stability index* of the periodic solution $\mathbf{z}(t, \mathbf{z}_0)$.

If solution $\mathbf{z}(t, \mathbf{z}_0)$ is singly g_i-symmetric with initial condition $\mathbf{z}_0 \in \Sigma_i$, then in a half of period $\mathbf{z}(T/2, z_0) \in \Sigma_i$, and monodromy matrices computed from the points $\mathbf{z}_0$ and $\mathbf{z}(T/2)$ are correspondingly

$$M_i = \widetilde{G}_i Z^{\mathrm{T}}(T/2) \widetilde{G}_i Z(T/2), \quad \widetilde{M}_i = Z(T/2) \widetilde{G}_i Z^{\mathrm{T}}(T/2) \widetilde{G}_i,$$

where $\widetilde{G}_i = G_i J$, $i = 1, 2$.

If solution $\mathbf{z}(t, \mathbf{z}_0)$ is doubly symmetric with initial condition $\mathbf{z}_0 \in \Sigma_i$, then in a quarter of the period $\mathbf{z}(T/4, \mathbf{z}_0) \in \Sigma_{3-i}$ and monodromy matrices computed from the points $\mathbf{z}_0$ and $\mathbf{z}(T/4)$ are

$$M_i = \left[\widetilde{G}_i Z^{\mathrm{T}}(T/4)\widetilde{G}_{3-i} Z(T/4)\right]^2, \quad \widetilde{M}_i = \left[Z(T/4)\widetilde{G}_i Z^{\mathrm{T}}(T/4)\widetilde{G}_{3-i}\right]^2.$$

Matrices M_i and $\widetilde{M}_i$ are similar but the similarity transformation is too awkward. The stability index of such solution can be computed by formula

$$S = 2\left(1 + 2Z_{32}Z_{14} + 2Z_{41}Z_{23} - 4Z_{11}Z_{33} + 2Z_{31}Z_{13} - 2Z_{42}Z_{24}\right)^2 - 1,$$

where Z_{ij} are components of matrix $Z(T/4, Z_0)$. So S gets its minimal value equals to 1. Special structure of monodromy matrix of symmetric solution yields to presence of inner symmetry of M:

$$m_{11} = m_{33}, m_{22} = m_{44}, m_{12} = -m_{43}, m_{14} = -m_{23}, m_{21} = -m_{34}, m_{32} = -m_{41}.$$

2. Bifurcations of periodic solutions

In [1] a linear transformation with symplectic and orthogonal matrix A was proposed which is completely defined by normalized vector $\mathcal{H}$ of the phase velocity $\mathbf{v}_1(\mathbf{z}_0)$: $\mathcal{H} = \mathbf{v}_1(\mathbf{z}_0)/|\mathbf{v}_1(\mathbf{z}_0)|$.

The bifurcation analysis of families of singly symmetric periodic solutions was provided earlier (see [2, 1]. Here we give such analysis for doubly symmetric solutions.

In the case $S = 1$ there are two possibilities. Either matrix M has only one elementary divisor $(\lambda - 1)^4$, or it has two elementary divisors $(\lambda - 1)^2$ and $(\lambda - 1)^2$. The first one corresponds to a fold (saddle-node) bifurcation of periodic solution at which the family riches the extrema of H. The second one corresponds to a pitch-fork bifurcation, where a pair of singly symmetric periodic solutions appears.

In the case $S = -1$ period doubling bifurcation takes place. For singly symmetric solution the matrix M has 2 elementary divisors $(\lambda - 1)^2$ and $(\lambda + 1)^2$. The eigenvector corresponding to the first elementary divisor gives the direction of continuation of the initial family of periodic solutions. The second eigenvector gives the direction of continuation of a new family of periodic solutions with period $T' = 2T$. This family has the extremum on value $H(Z)$ and preserves the type of symmetry of the initial family.

Let consider the case of doubly symmetric periodic solution.

Statement 1. *The monodromy matrix M of doubly periodic solution in the case $S = -1$ always has 3 elementary divisors $(\lambda - 1)^2$, $\lambda + 1$ and $\lambda + 1$. Eigenvector, corresponding to the first elementary divisor gives the continuation of initial family. Eigenvectors, corresponding to the divisors $\lambda + 1$, give the continuation of the family of double periodic solutions but singly symmetric each. These double periodic solutions have different types of symmetry. The families of double periodic solutions* *rich the extremum of $H(\mathbf{z})$ at the bifurcation point. There are two scenarios of period doubling bifurcation in the case of doubly periodic solutions.*

- *Each of two new families has the same type of extremum at the bifurcation point (minimum or maximum). In this case both new families of singly symmetric solutions with period $T' = 2T$ exist near the initial family of doubly symmetric solutions.*
- *Two new families has different types of extremum and thus at each value of the family parameter h there exist the initial family of doubly symmetric solutions and only one new family of singly symmetric solutions with certain type of symmetry.*

Both these scenarios were investigated for periodic solutions of the Hill problem [3] and for periodic solutions of the restricted three body problem in the case of equal masses. Some new families of periodic solutions were found and were studied as well.

The previous situation is a special case of period multiplying bifurcation.

Statement 2. *Let doubly symmetric periodic solution with period T has the stability index $S = \cos 2\pi p/q$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$.*

- *If both p and q are odd, then there exists in vicinity of initial solution one family of doubly symmetric periodic solutions with period $T' = qT$;*
- *If at least one of the numbers is even, then there exist four families mutually pairwise symmetric singly symmetric periodic solutions with period $T' = qT$.*

In all cases except the case $p/q = 1/3$, new families reach the extremum on $H(\mathbf{z})$ at the branching point.

References

[1] B. B. Kreisman. Families of periodic solutions to hamiltonian systems: nonsymmetrical periodic solutions for a planar restricted three-body problem. *Cosmic Research*, 43(2):84–106, 2005.

[2] Alexander D. Bruno. *The Restricted 3–body Problem: Plane Periodic Orbits.* Walter de Gruyter, Berlin, 1994.

[3] Alexander. B. Batkhin and Natalia. V. Batkhina. *Hill's Problem.* Volgogradskoe nauchnoe izdatel'stvo, Volgograd, 2009. (in Russian).

Alexander Batkhin
Department of Singular Problems
Keldysh Institute of Applied Mathematics of RAS
Department of Theoretical Mechanics
Moscow Institute of Physics and Technology
Moscow, Russia
e-mail: batkhin@gmail.com