Some properties of doubly symmetric periodic solutions to Hamiltonian system

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Introduction I

Consider a time-independent non-integrable Hamiltonian system with two degrees of freedom, which canonical equations of motion posses the only first integral $H(\mathbf{z}) = h$, where $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathbb{M} \equiv \mathbb{R}^4$ and $H(\mathbf{z})$ is a smooth Hamiltonian function of a system.

Let the equations of motion

$$\dot{\mathbf{z}} = J \operatorname{grad} H(\mathbf{z}), \text{ here } J = \begin{pmatrix} \mathbf{0} & \mathbf{E}^2 \\ -\mathbf{E}^2 & \mathbf{0} \end{pmatrix}, \tag{1}$$

are invariant under the discrete group $\mathcal{G} \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$ of linear automorphisms of the extended phase space $\mathbb{R} \times \mathbb{M}$.

Introduction II

Two generators g_1, g_2 of the group $\mathcal{G}$ are involutive operators, i.e. $g_i^2 = \mathrm{id}$, the third non-trivial transformation $g_3 = g_1 \circ g_2 = g_2 \circ g_1$. Then each solution $\mathbf{z}(t, \mathbf{z}_0)$ to equation (1) with initial condition $\mathbf{z}(0) = \mathbf{z}_0$ belongs to one of the following group depending on the type of symmetry.

- **Non-symmetric** solutions, which change under any automorphism g_i,$i = 1, 2, 3$.
- **Singly symmetric** solutions, which are invariant under only one automorphism g_i,$i = 1, 2, 3$.
- **Doubly symmetric** solutions, which are invariant under any automorphism g_i, $i = 1, 2, 3$.

Introduction III

Let $\mathbf{z}(t, \mathbf{z}_0)$ be a periodic solution with period T. In the case of time-independent system solutions $\mathbf{z}(t, \mathbf{z}_0)$ belongs to a family of one-parametric periodic solutions. The parameter of such family is a value of the first integral $H(\mathbf{z})$ of system (1).
Let consider that the phase coordinates $\mathbf{z}$ are chosen in a such manner that the involutive transformation g_i has the form $g_i : (t, \mathbf{z}) \to (\sigma t, G_i \mathbf{z})$, where $\sigma = \pm 1$, G_i is a constant matrix. Let $\Sigma_i \equiv \{\mathbf{z} | g_i(\mathbf{z}) = \mathbf{z}\}$ is an invariant set of the transformation g_i. Then there exist g_i-invariant periodic solutions, which are completely defined by a part of their phase trajectories contained between points $\mathbf{z}_0$ and $\mathbf{z}(T/2)$ lying on the Σ_i.

The goal

To study in linear approximation the dynamics near doubly symmetric periodic solution $\mathbf{z}(t, \mathbf{z}_0)$ and provide its bifurcation analysis as well.

Properties of doubly symmetric solution I

Dynamics of the system (1) in the vicinity of a solution $\mathbf{z}(t)$ is described in linear approximation by matrix $Z(t, \mathbf{z})$, which is the solution to the Cauchy problem of the Poincare variational equation

$$\dot{Z} = J \operatorname{Hess} H(\mathbf{z}) Z, \quad Z_0 = E^4,$$

where $\operatorname{Hess} H(\mathbf{z})$ is the Hessian of function H computed along the solution $\mathbf{z}(t, \mathbf{z}_0)$. Matrix $Z(t)$ is symplectic: $Z^T J Z = J$ and its characteristic polynomial $P(\lambda)$ is reciprocal. Monodromy matrix M of periodic solution $\mathbf{z}(t, \mathbf{z}_0)$ is $Z(T, Z_0)$ with a property $Z(t + T) = Z(t)M$. Eigenvalues ρ_k of matrix M are called ***multiplicators***.

Properties of doubly symmetric solution II

Monodromy matrix M has the following properties:

- multiplicators ρ_k are mutually complex conjugate and mutually inverse;
- matrix M has eigenvector $\mathbf{v}_1 \equiv J \operatorname{grad} H(\mathbf{z}_0)$, corresponding to multiplicator $\rho_{1,2} = 1$ with multiplicity 2;
- characteristic polynomial $P(\lambda)$ of matrix M is factorized

$$P(\lambda) = (\lambda - 1)^2 \left(\lambda^2 - 2S\lambda + 1\right),$$

where S is ***stability index*** of the periodic solution $\mathbf{z}(t, \mathbf{z}_0)$.

Properties of doubly symmetric solution III

If solution $\mathbf{z}(t, \mathbf{z}_0)$ is doubly symmetric with initial condition $\mathbf{z}_0 \in \Sigma_i$, then in a quarter of the period $\mathbf{z}(T/4, \mathbf{z}_0) \in \Sigma_{3-i}$ and monodromy matrices computed from the points $\mathbf{z}_0$ and $\mathbf{z}(T/4)$ are

$$M_i = \left[\widetilde{G}_i Z^{\mathrm{T}}(T/4)\widetilde{G}_{3-i} Z(T/4)\right]^2, \quad \widetilde{M}_i = \left[Z(T/4)\widetilde{G}_i Z^{\mathrm{T}}(T/4)\widetilde{G}_{3-i}\right]^2.$$

Matrices M_i and $\widetilde{M}_i$ are similar but the similarity transformation is too awkward. The stability index of such solution can be computed by formula

$$S = 2\left(1 + 2Z_{32}Z_{14} + 2Z_{41}Z_{23} - 4Z_{11}Z_{33} + 2Z_{31}Z_{13} - 2Z_{42}Z_{24}\right)^2 - 1,$$

where Z_{ij} are components of matrix $Z(T/4, Z_0)$. So S gets its minimal value equals to 1.

Properties of doubly symmetric solution IV

Special structure of monodromy matrix of symmetric solution yields to presence of inner symmetry of M:

$$m_{11} = m_{33}, m_{22} = m_{44}, m_{12} = -m_{43},$$
$$m_{14} = -m_{23}, m_{21} = -m_{34}, m_{32} = -m_{41}.$$

Bifurcations of periodic solutions I

In [5] a linear transformation with symplectic and orthogonal matrix A_i was proposed which is completely defined by normalized vector $\mathcal{H}$ of the phase velocity $\mathbf{v}_1(\mathbf{z}_0)$:

$$\mathcal{H} = \mathbf{v}_1(\mathbf{z}_0)/|\mathbf{v}_1(\mathbf{z}_0)|.$$

If $\mathbf{z}_0 \in \Sigma_i$, $i = 1, 2$, then A_i is the following

$$A_1 = \begin{pmatrix} 0 & -\mathcal{H}_4 & \mathcal{H}_1 & 0 \\ \mathcal{H}_4 & 0 & 0 & -\mathcal{H}_1 \\ -\mathcal{H}_1 & 0 & 0 & -\mathcal{H}_4 \\ 0 & \mathcal{H}_1 & \mathcal{H}_4 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} \mathcal{H}_3 & 0 & 0 & \mathcal{H}_2 \\ 0 & \mathcal{H}_3 & \mathcal{H}_2 & 0 \\ 0 & -\mathcal{H}_2 & \mathcal{H}_3 & 0 \\ -\mathcal{H}_2 & 0 & 0 & \mathcal{H}_3 \end{pmatrix}$$

Bifurcations of periodic solutions II

Then matrix $N = A^{\mathrm{T}} M A$ obtained after the transformation is the following

$$N = \begin{pmatrix} 1 & n_{12} & n_{13} & n_{14} \\ 0 & n_{22} & -n_{14} & n_{24} \\ 0 & 0 & 1 & 0 \\ 0 & n_{42} & -n_{12} & n_{22} \end{pmatrix}.$$

Stability index $S = n_{22}$.

The bifurcation analysis of families of singly symmetric periodic solutions was provided earlier (see [3, 5]). Here we give such analysis for doubly symmetric solutions.

Bifurcations in case $S = 1$

There are two possibilities:

- Either M has only one elementary divisor $(\lambda - 1)^4$ and a saddle-node bifurcation takes place, at which the family riches the extrema of H.
- M has two elementary divisors $(\lambda - 1)^2$ and $(\lambda - 1)^2$ and a pitch-fork bifurcation takes place, where a pair of singly symmetric periodic solutions appears.

Bifurcations in case $S = -1$

Statement 1

Matrix M of doubly periodic solution always has 3 elementary divisors $(\lambda - 1)^2$, $\lambda + 1$ and $\lambda + 1$. Eigenvectors, corresponding to the divisors $\lambda + 1$, give the continuation of the family of double periodic solutions but singly symmetric each. These double periodic solutions have different types of symmetry. There are two scenarios of period doubling bifurcation.

- Each of two new families has the same type of extremum at the bifurcation point (minimum or maximum). In this case both new families of singly symmetric solutions with period $T' = 2T$ exist near the initial family of doubly symmetric solutions.
- Two new families has different types of extremum and thus at each value of the family parameter h there exist the initial family of doubly symmetric solutions and only one new family of singly symmetric solutions with certain type of symmetry.

Period multiplying bifurcation

Statement 2

Let doubly symmetric periodic solution with period T has the stability index $S = \cos 2\pi p/q$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

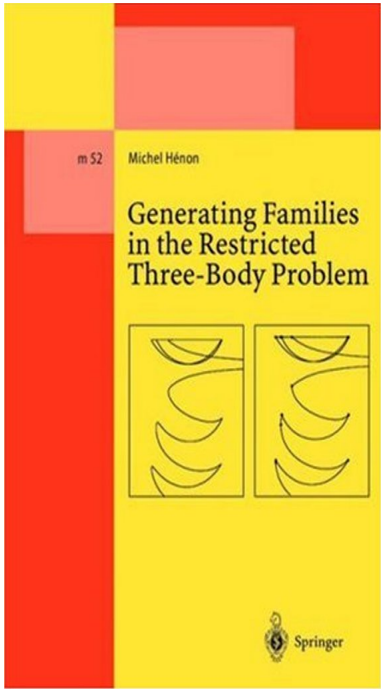
- If both p and q are odd, then there exists in vicinity of initial solution one family of doubly symmetric periodic solutions with period $T' = qT$;
- If at least one of the numbers is even, then there exist four families mutually pairwise symmetric singly symmetric periodic solutions with period $T' = qT$.

In all cases except the case $p : q = 1 : 3$, new families reach the extremum on $H(\mathbf{z})$ at the branching point.

Hill problem I

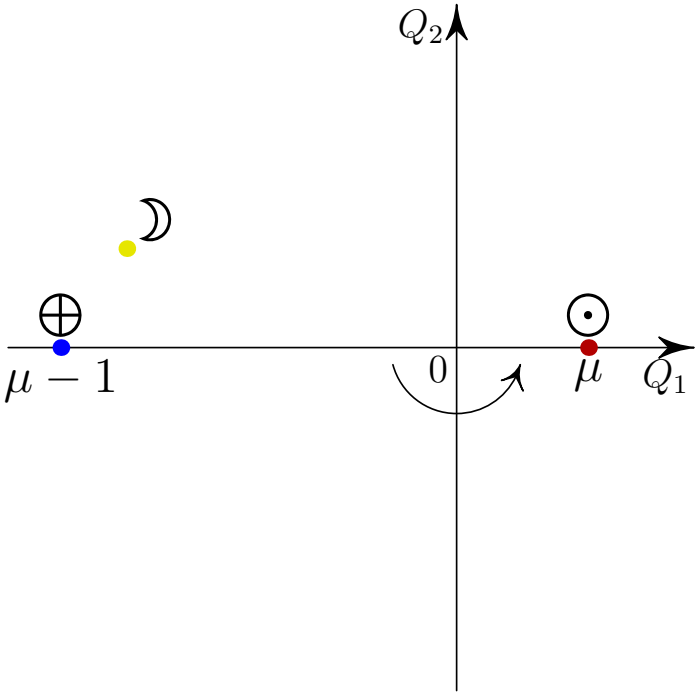
Michel Hénon "Generating families in the restricted three-body problem" [4]

"...Hill's problem, which is non-trivial, non-integrable dynamical system; they cannot be analyzed in terms of two-body problem, and can in fact be obtained only through numerical investigation..."

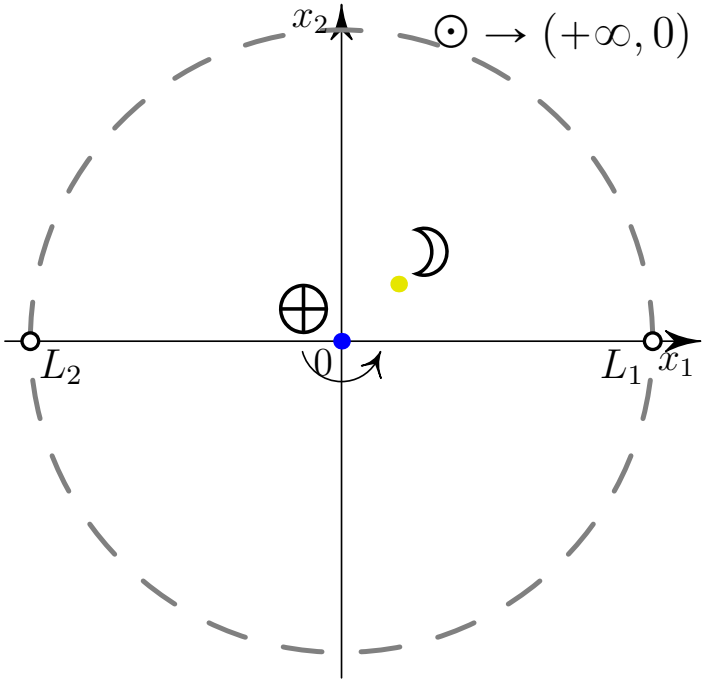


Hill problem II

Planar ***Hill problem (HP)*** is a celestial mechanics model being a limit case of the well known restricted three body problem.



Restricted 3-body problem



Hill problem

Hill problem Hamiltonian function I

Hill problem Hamiltonian function

$$H = \frac{1}{2}\left(y_1^2 + y_2^2\right) + x_2 y_1 - x_1 y_2 + \frac{1}{2}\left(x_1^2 + x_2^2\right) - \frac{3}{2}x_1^2 - \frac{1}{r}$$

- Kinetic energy
- Potential of Coriolis forces
- Potential of centrifugal force
- Gravitational potential of the Sun
- Potential of the central body

Hill problem properties

The essential property of the HP is the presence of two symmetries of extended phase space given by linear transformations

$$\begin{aligned}
\Sigma_1 &: (t, x_1, x_2, y_1, y_2) \to (-t, x_1, -x_2, -y_1, y_2) \\
\Sigma_2 &: (t, x_1, x_2, y_1, y_2) \to (-t, -x_1, x_2, y_1, -y_2)\,, \\
\Sigma_{12} \equiv \Sigma_1 \circ \Sigma_2 &: (t, x_1, x_2, x_1, x_2) \to (t, -x_1, -x_2, -y_1, -y_2)
\end{aligned}$$

which involves that all the periodic solutions of the HP belong to one of the following group:

1. Asymmetric solutions, which change their form under any transformation $\Sigma_{1,2}$.
2. Single symmetric solutions, which are invariant under only one transformation Σ_1 or Σ_2.
3. Double symmetric solutions, which are invariant under any transformation $\Sigma_{1,2}$.

Bifurcation analysis of families of doubly symmetric solutions I

HP provides us with infinite number of periodic solution. We investigate some families of doubly symmetric solutions, which stability index S comes through the interval $[-1; +1]$.

Example of family f_3

f_3 is a family of doubly symmetric periodic solutions, which was found out and investigated numerically by Prof. M. Hénon.

Bifurcation analysis of families of doubly symmetric solutions II

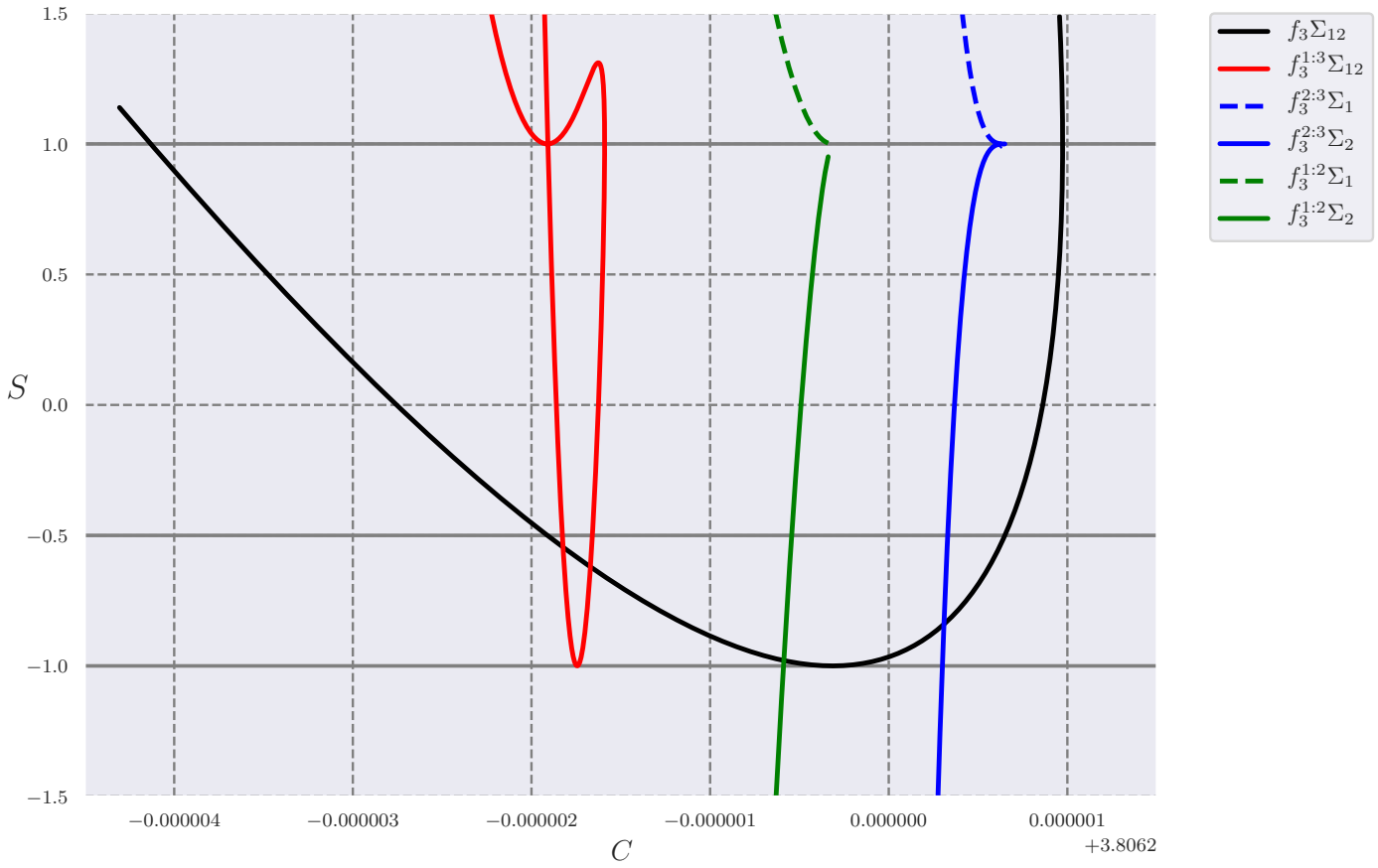


Figure 1: Characteristic of family f_3 and families from period multiplying bifurcations.

Bifurcation analysis of families of doubly symmetric solutions III

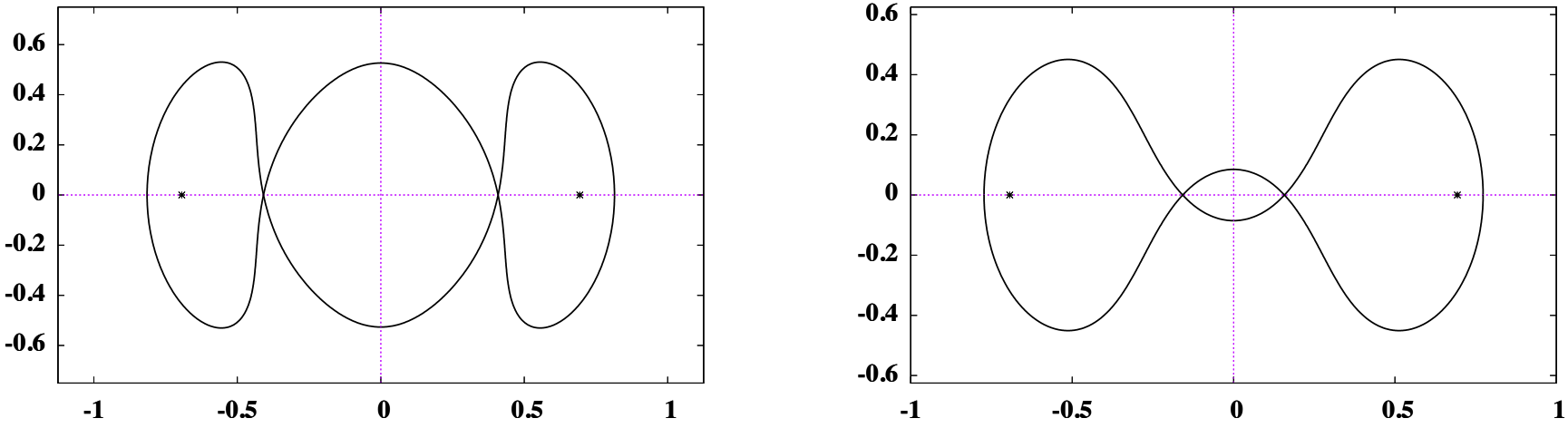


Figure 2: Examples of f_3 family doubly symmetric orbits.

Bifurcation analysis of families of doubly symmetric solutions IV

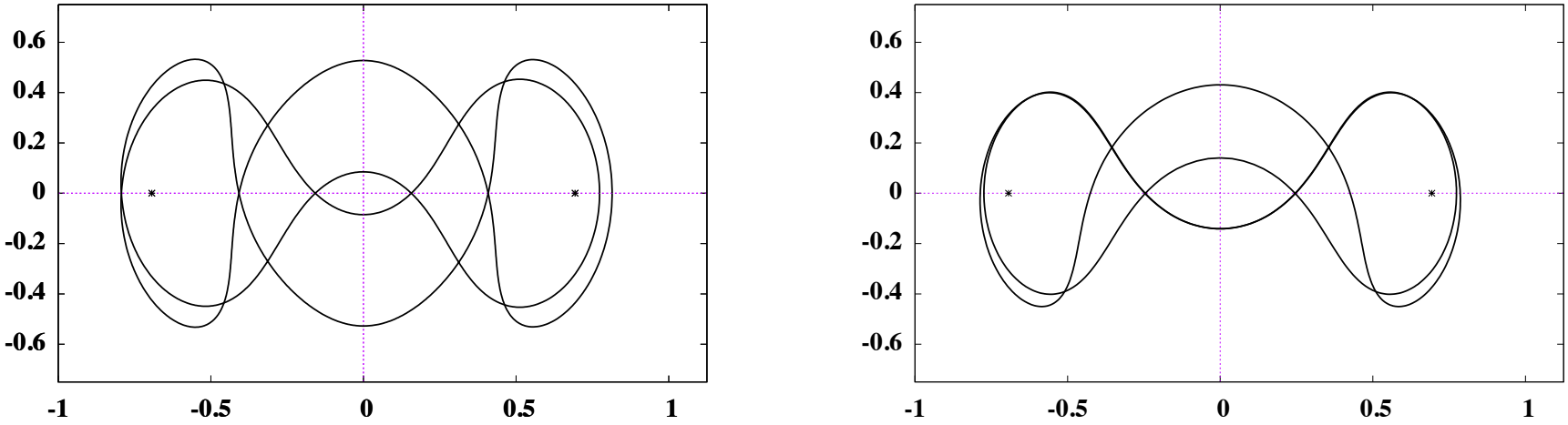


Figure 3: Continuation of double periodic singly symmetric orbits: Σ_1-symmetric (left) and Σ_2-symmetric (right).

Bifurcations of the Hill problem periodic solutions

All the scenarios described above were investigated for periodic solutions of the Hill problem [2]. Some new families of periodic solutions were found and studied. In particularly for family f_3 [1] all types of mentioned above bifurcations where discovered for resonances $p/q \in \{1:1, 1:3, 1:2, 2:3\}$.

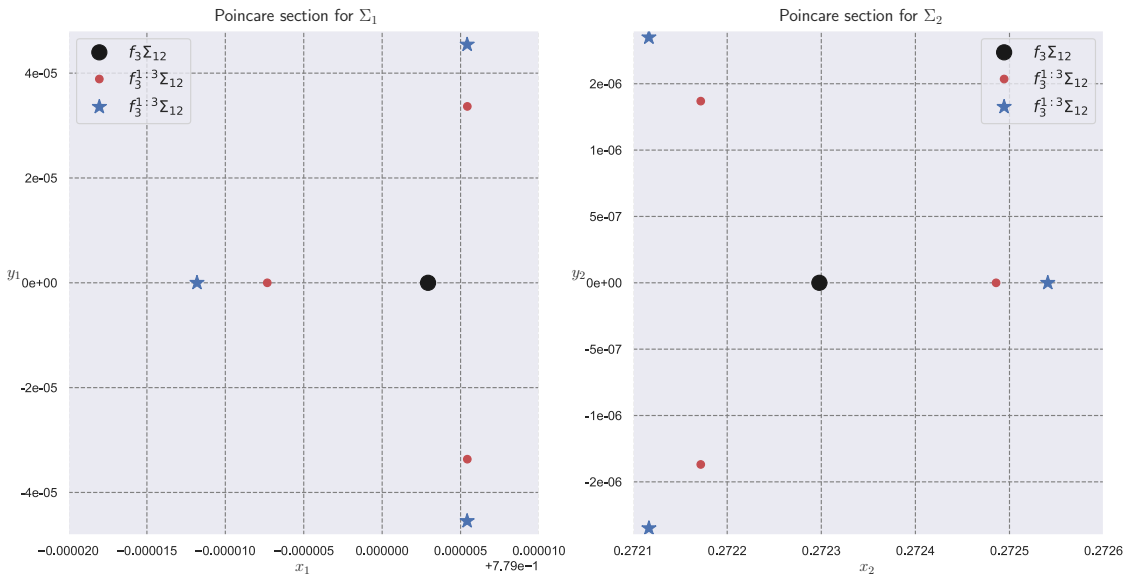


Figure 4: Poincaré sections by planes Σ_1 (left) and Σ_2 (right) with doubly symmetric resonances $1:3$ of doubly symmetric solution f_3 are shown.

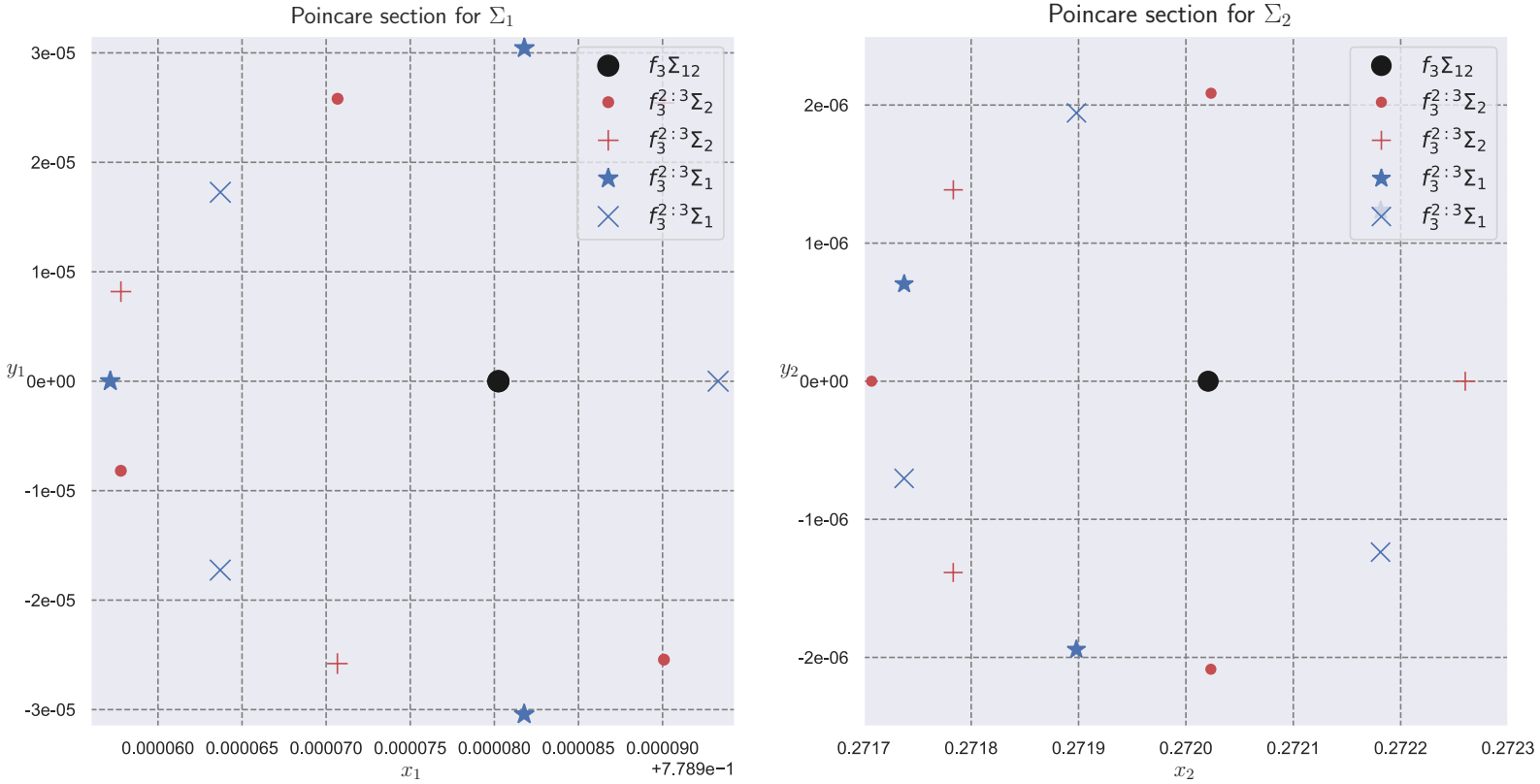


Figure 5: Poincaré sections by planes Σ_1 (left) and Σ_2 (right) singly symmetric with resonances $2:3$ of doubly symmetric solution f_3 are shown.

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