Some properties of doubly symmetric periodic solutions to Hamiltonian system

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Consider a time-independent non-integrable Hamiltonian system with two degrees of freedom, which canonical equations of motion posses the only first integral $H(z) = h$, where $z = (x, y) \in M \equiv \mathbb{R}^4$ and $H(z)$ is a smooth Hamiltonian function of a system.

Let the equations of motion

$$\dot{z} = J \text{grad} H(z), \quad \text{here} \quad J = \begin{pmatrix} 0 & E^2 \\ -E^2 & 0 \end{pmatrix},$$

are invariant under the discrete group $G \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$ of linear automorphisms of the extended phase space $\mathbb{R} \times M$. 
Two generators $g_1, g_2$ of the group $\mathcal{G}$ are involutive operators, i.e. $g_i^2 = \text{id}$, the third non-trivial transformation $g_3 = g_1 \circ g_2 = g_2 \circ g_1$. Then each solution $z(t, z_0)$ to equation (1) with initial condition $z(0) = z_0$ belongs to one of the following group depending on the type of symmetry.

- **Non-symmetric** solutions, which change under any automorphism $g_i, i = 1, 2, 3$.

- **Singly symmetric** solutions, which are invariant under only one automorphism $g_i, i = 1, 2, 3$.

- **Doubly symmetric** solutions, which are invariant under any automorphism $g_i, i = 1, 2, 3$. 
Introduction III

Let $z(t, z_0)$ be a periodic solution with period $T$. In the case of time-independent system solutions $z(t, z_0)$ belongs to a family of one-parametric periodic solutions. The parameter of such family is a value of the first integral $H(z)$ of system (1).

Let consider that the phase coordinates $z$ are chosen in a such manner that the involutive transformation $g_i$ has the form $g_i : (t, z) \to (\sigma t, G_i z)$, where $\sigma = \pm 1$, $G_i$ is a constant matrix. Let $\Sigma_i \equiv \{ z | g_i(z) = z \}$ is an invariant set of the transformation $g_i$. Then there exist $g_i$-invariant periodic solutions, which are completely defined by a part of their phase trajectories contained between points $z_0$ and $z(T/2)$ lying on the $\Sigma_i$.

The goal

To study in linear approximation the dynamics near doubly symmetric periodic solution $z(t, z_0)$ and provide its bifurcation analysis as well.
Properties of doubly symmetric solution

Dynamics of the system (1) in the vicinity of a solution $z(t)$ is described in linear approximation by matrix $Z(t, z)$, which is the solution to the Cauchy problem of the Poincare variational equation

$$\dot{Z} = J\text{Hess } H(z) Z, \quad Z_0 = E^4,$$

where $\text{Hess } H(z)$ is the Hessian of function $H$ computed along the solution $z(t, z_0)$. Matrix $Z(t)$ is symplectic: $Z^T J Z = J$ and its characteristic polynomial $P(\lambda)$ is reciprocal. Monodromy matrix $M$ of periodic solution $z(t, z_0)$ is $Z(T, Z_0)$ with a property $Z(t + T) = Z(t)M$. Eigenvalues $\rho_k$ of matrix $M$ are called *multiplicators*. 
Properties of doubly symmetric solution II

Monodromy matrix $M$ has the following properties:

- multiplicators $\rho_k$ are mutually complex conjugate and mutually inverse;
- matrix $M$ has eigenvector $\mathbf{v}_1 \equiv J \text{grad } H(z_0)$, corresponding to multiplicantor $\rho_{1,2} = 1$ with multiplicity 2;
- characteristic polynomial $P(\lambda)$ of matrix $M$ is factorized

$$P(\lambda) = (\lambda - 1)^2 (\lambda^2 - 2S\lambda + 1),$$

where $S$ is stability index of the periodic solution $z(t, z_0)$. 


Properties of doubly symmetric solution III

If solution $z(t, z_0)$ is doubly symmetric with initial condition $z_0 \in \Sigma_i$, then in a quarter of the period $z(T/4, z_0) \in \Sigma_{3-i}$ and monodromy matrices computed from the points $z_0$ and $z(T/4)$ are

$$M_i = \left[ \tilde{G}_i Z^T (T/4) \tilde{G}_{3-i} Z(T/4) \right]^2, \quad \tilde{M}_i = \left[ Z(T/4) \tilde{G}_i Z^T (T/4) \tilde{G}_{3-i} \right]^2. $$

Matrices $M_i$ and $\tilde{M}_i$ are similar but the similarity transformation is too awkward. The stability index of such solution can be computed by formula

$$S = 2(1 + 2Z_{32}Z_{14} + 2Z_{41}Z_{23} - 4Z_{11}Z_{33} + 2Z_{31}Z_{13} - 2Z_{42}Z_{24})^2 - 1,$$

where $Z_{ij}$ are components of matrix $Z(T/4, z_0)$. So $S$ gets its minimal value equals to 1.
Special structure of monodromy matrix of symmetric solution yields to presence of inner symmetry of $M$:

\[
\begin{align*}
m_{11} &= m_{33}, \quad m_{22} = m_{44}, \quad m_{12} = -m_{43}, \\
m_{14} &= -m_{23}, \quad m_{21} = -m_{34}, \quad m_{32} = -m_{41}.
\end{align*}
\]
In [5] a linear transformation with symplectic and orthogonal matrix $A_i$ was proposed which is completely defined by normalized vector $\mathcal{H}$ of the phase velocity $v_1(z_0)$:

$$\mathcal{H} = v_1(z_0)/|v_1(z_0)|.$$

If $z_0 \in \Sigma_i, i = 1, 2$, then $A_i$ is the following

$$A_1 = \begin{pmatrix} 0 & -\mathcal{H}_4 & \mathcal{H}_1 & 0 \\
\mathcal{H}_4 & 0 & 0 & -\mathcal{H}_1 \\
-\mathcal{H}_1 & 0 & 0 & -\mathcal{H}_4 \\
0 & \mathcal{H}_1 & \mathcal{H}_4 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} \mathcal{H}_3 & 0 & 0 & \mathcal{H}_2 \\
0 & \mathcal{H}_3 & \mathcal{H}_2 & 0 \\
0 & -\mathcal{H}_2 & \mathcal{H}_3 & 0 \\
-\mathcal{H}_2 & 0 & 0 & \mathcal{H}_3 \end{pmatrix}$$
Bifurcations of periodic solutions II

Then matrix $N = A^T M A$ obtained after the transformation is the following

$$N = \begin{pmatrix}
1 & n_{12} & n_{13} & n_{14} \\
0 & n_{22} & -n_{14} & n_{24} \\
0 & 0 & 1 & 0 \\
0 & n_{42} & -n_{12} & n_{22}
\end{pmatrix}.$$ 

Stability index $S = n_{22}$.

The bifurcation analysis of families of singly symmetric periodic solutions was provided earlier (see [3, 5]). Here we give such analysis for doubly symmetric solutions.
Bifurcations in case $S = 1$

There are two possibilities:

- Either $\mathcal{M}$ has only one elementary divisor $(\lambda - 1)^4$ and a saddle-node bifurcation takes place, at which the family riches the extrema of $H$.
- $\mathcal{M}$ has two elementary divisors $(\lambda - 1)^2$ and $(\lambda - 1)^2$ and a pitch-fork bifurcation takes place, where a pair of singly symmetric periodic solutions appears.
Bifurcations in case $S = -1$

Statement 1

Matrix $M$ of doubly periodic solution always has 3 elementary divisors $(\lambda - 1)^2$, $\lambda + 1$ and $\lambda + 1$. Eigenvectors, corresponding to the divisors $\lambda + 1$, give the continuation of the family of double periodic solutions but singly symmetric each. These double periodic solutions have different types of symmetry. There are two scenarios of period doubling bifurcation.

- Each of two new families has the same type of extremum at the bifurcation point (minimum or maximum). In this case both new families of singly symmetric solutions with period $T' = 2T$ exist near the initial family of doubly symmetric solutions.

- Two new families has different types of extremum and thus at each value of the family parameter $h$ there exist the initial family of doubly symmetric solutions and only one new family of singly symmetric solutions with certain type of symmetry.
**Period multiplying bifurcation**

**Statement 2**

Let doubly symmetric periodic solution with period $T$ has the stability index $S = \cos \frac{2\pi p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

- If both $p$ and $q$ are odd, then there exists in vicinity of initial solution one family of doubly symmetric periodic solutions with period $T' = qT$;
- If at least one of the numbers is even, then there exist four families mutually pairwise symmetric singly symmetric periodic solutions with period $T' = qT$.

In all cases except the case $p : q = 1 : 3$, new families reach the extremum on $H(z)$ at the branching point.
Michel Hénon “Generating families in the restricted three-body problem” [4]

“...Hill’s problem, which is non-trivial, non-integrable dynamical system; they cannot be analyzed in terms of two-body problem, and can in fact be obtained only through numerical investigation...”
Hill problem II

Planar *Hill problem (HP)* is a celestial mechanics model being a limit case of the well known restricted three body problem.

- **Restricted 3-body problem**
- **Hill problem**
Hill problem Hamiltonian function

\[ H = \frac{1}{2} \left( y_1^2 + y_2^2 \right) + x_2 y_1 - x_1 y_2 + \frac{1}{2} \left( x_1^2 + x_2^2 \right) - \frac{3}{2} x_1^2 - \frac{1}{r} \]

- Kinetic energy
- Potential of Coriolis forces
- Potential of centrifugal force
- Gravitational potential of the Sun
- Potential of the central body
Hill problem properties

The essential property of the HP is the presence of two symmetries of extended phase space given by linear transformations

\[ \Sigma_1 : (t, x_1, x_2, y_1, y_2) \rightarrow (-t, x_1, -x_2, -y_1, y_2) \]
\[ \Sigma_2 : (t, x_1, x_2, y_1, y_2) \rightarrow (-t, -x_1, x_2, y_1, -y_2) \]
\[ \Sigma_{12} \equiv \Sigma_1 \circ \Sigma_2 : (t, x_1, x_2, x_1, x_2) \rightarrow (t, -x_1, -x_2, -y_1, -y_2) \]

which involves that all the periodic solutions of the HP belong to one of the following group:

1. Asymmetric solutions, which change their form under any transformation \( \Sigma_{1,2} \).
2. Single symmetric solutions, which are invariant under only one transformation \( \Sigma_1 \) or \( \Sigma_2 \).
3. Double symmetric solutions, which are invariant under any transformation \( \Sigma_{1,2} \).
Bifurcation analysis of families of doubly symmetric solutions I

HP provides us with infinite number of periodic solution. We investigate some families of doubly symmetric solutions, which stability index $S$ comes through the interval $[-1; +1]$.

**Example of family $f_3$**

$f_3$ is a family of doubly symmetric periodic solutions, which was found out and investigated numerically by Prof. M. Hénon.
**Figure 1:** Characteristic of family $f_3$ and families from period multiplying bifurcations.
Bifurcation analysis of families of doubly symmetric solutions III

Figure 2: Examples of $f_3$ family doubly symmetric orbits.
Figure 3: Continuation of double periodic singly symmetric orbits: $\Sigma_1$-symmetric (left) and $\Sigma_2$-symmetric (right).
Bifurcations of the Hill problem periodic solutions

All the scenarios described above were investigated for periodic solutions of the Hill problem [2]. Some new families of periodic solutions were found and studied. In particularly for family $f_3$ [1] all types of mentioned above bifurcations where discovered for resonances $p/q \in \{1 : 1, 1 : 3, 1 : 2, 2 : 3\}$.

**Figure 4:** Poincaré sections by planes $\Sigma_1$ (left) and $\Sigma_2$ (right) with doubly symmetric resonances $1 : 3$ of doubly symmetric solution $f_3$ are shown.
Figure 5: Poincaré sections by planes $\Sigma_1$ (left) and $\Sigma_2$ (right) singly symmetric with resonances 2 : 3 of doubly symmetric solution $f_3$ are shown.
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