Galois elliptic function and its symmetries

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Abstract. The vast subject of elliptic integrals, functions and curves has numerous applications in physics. Two distinct approaches to investigating elliptic functions have become “classical”: that of Carl Jacobi and that of Karl Weierstrass. Two distinct chapters are dedicated to these two approaches in Whittaker and Watson (famous) “Course of Modern Analysis” without attempts of unification. Some have thus claimed that the Weierstrass approach is more suitable for “theoretical” research, whereas the Jacobi elliptic functions arise “more frequently” in applications. Yet and indeed such dichotomy is artificial and the study of elliptic functions and curves can and must be naturally united via an algebraic approach, readily producing a most canonical “essential” elliptic function which preserving transformations acquire “simplest” forms. Although such “natural” building block, to which we (justifiably) ought to refer as *the Galois elliptic function*, has been only recently introduced, its exploitation has already been quite fruitful. Not merely for efficiently regenerating known and established results but for attaining new calculations which previously seemed too cumbersome to pursue via either the Jacobi or the Weierstrass approach. Here, at PCA 2019, we aim at demonstrating the methodological significance of this naturally algebraic approach via applying it to a few but quite fundamental problems of classical mechanics, thereby producing formidable, hardly standard and highly efficient solutions!

1. Preliminary definitions

For each nonzero c, define a *homothety operator* $H(c)$ via its action on an arbitrary function f:

$$H(c)f(x) := cf\left(x/c\right).$$

Thus, the graph of the function f when subjected to homothety, with ratio c, with the origin being the fixed point, yields the graph of the function $H(c)f$.
We shall use an upper subscript to indicate the functional transformation

$$f^c(x) := cf\left(\sqrt{c}x\right),$$

which we shall not hesitate to explicitly rewrite whenever a risk of confusing such a notation, with the common use of upper subscripts for denoting powers, arises. While keeping in mind that the square root is not single valued, we shall assume, unless we indicate otherwise, its values to lie in the right half plane without the imaginary negative ray. With this choice of branch for the square root one guarantees that

$$\sqrt{\bar{x}} = \overline{\sqrt{x}},$$

with the bar denoting complex conjugation.
Let $\mathcal{H}$ denote the upper half of the complex plane

$$\mathcal{H} := \{z : \operatorname{Im}(z) > 0\}.$$

The group $PSL(2,\mathbb{R})$ acts faithfully on $\mathcal{H}$, and constitutes the group of its conformal automorphisms.
Let $\mathcal{D}$ denote the fundamental domain for the action of the modular group $PSL(2,\mathbb{Z})$, being a discrete subgroup of $PSL(2,\mathbb{R})$, upon $\mathcal{H}$

$$\mathcal{D} := \mathcal{H}/PSL(2,\mathbb{Z}).$$

Denoting by $\mathcal{L}$ the set of lattices, that is the set of discrete subgroups of rank 2, in $\mathbb{C}$, we may identify $\mathcal{D}$ with the quotient $\mathcal{L}/\mathbb{C}^{\times}$, that is, the quotient of the action of the multiplicative group of invertible elements of $\mathbb{C}$ upon $\mathcal{L}$.
The Eisenstein series of index k is a function G_k, on the upper half plane $\mathcal{H}$, determined by the equality

$$G_k(x) := \sum_{(n,m)\in\mathbb{Z}\oplus\mathbb{Z}\setminus(0,0)} (m\,x+n)^{-2k}.$$

When the index k is an integer strictly exceeding 1, G_k is a modular form of weight k. In other words, it is then a holomorphic function on the extended upper half plane $\mathcal{H}\cup\infty$, satisfying the relation

$$G_k\left(\frac{ax+b}{cx+d}\right) = (cx+d)^{2k}G_k(x) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Exploiting the identity

$$G_k(\infty) = 2\zeta(2k),$$

where ζ is the Riemann ζ-function and setting

$$\Delta := 4h_2^3 - 27h_3^2, \quad h_2 = 15G_2,\ h_3 = 35G_3,$$

one finds that

$$\Delta(\infty) = 4\left(\frac{\pi^4}{3}\right)^3 - 27\left(\frac{2\pi^6}{27}\right)^2 = 0.$$

Thereby Δ is a cusp form of weight 6.
The Klein j-invariant is defined as

$$j := \frac{4h_2^3}{\Delta} \tag{1}$$

and can be viewed, being a modular function of weight 0, as a bijection from the extended fundamental domain $\mathcal{D}\cup\infty$ onto the Riemann sphere $\mathbb{C}\cup\infty$ with a (simple) pole at infinity. The j-invariant is evidently fixed under the transformation

$$(h_2, h_3) \mapsto (c^2 h_2, c^3 h_3).$$

For an arbitrary pair (h_2, h_3), subject to the condition that Δ does not vanish, *the Weierstrass elliptic function* $\wp$ is the solution of the differential equation

$$y'^2 = 4\left(y^3 - h_2 y - h_3\right) = 4(y-e_1)(y-e_2)(y-e_3) \tag{2}$$

with a (double) pole at zero. The function $\wp$ represents a one parameter family of Weierstrass elliptic functions

$$\mathcal{F}_\wp := \{\wp^c :\ x \mapsto c\,\wp(\sqrt{c}\,x),\ c \in \mathbb{C}^\times\}. \tag{3}$$

Each $\wp^c \in \mathcal{F}_\wp$ satisfies a corresponding differential equation

$$y'^2 = 4\left(y^3 - c^2 h_2 y - c^3 h_3\right) = 4(y-ce_1)(y-ce_2)(y-ce_3), \tag{4}$$

and the whole family $\mathcal{F}_\wp$ shares a "common" j-invariant $j(\tau)$, corresponding to a single point τ in $\mathcal{D}$.

For a fixed k, we define *the Jacobi elliptic function* sn as the vanishing at zero solution of the differential equation

$$y'^2 = \left(1-y^2\right)\left(1-k^2y^2\right),$$

whose leading Maclaurin series coefficient is 1. The constant k is called *the elliptic modulus.*

Every Jacobi elliptic function sn represents a one parameter family of elliptic functions

$$\mathcal{F}_{\mathrm{sn}} := \{H\left(\sqrt{c}\,\right)\mathrm{sn} :\ c \in \mathbb{C}^\times\}. \tag{5}$$

Each $H\left(\sqrt{c}\,\right)\mathrm{sn} \in \mathcal{F}_{\mathrm{sn}}$ satisfies a corresponding differential equation

$$y'^2 = \left(1-y^2/c\right)\left(1-k^2y^2/c\right).$$

2. The Galois essential elliptic function

Let α be a parameter, $\alpha \in \mathbb{C}\setminus\{-2/3, 2/3\}$. Define an *essential elliptic function* $\mathcal{R}_\alpha$ as the solution of the differential equation

$$y'^2 = 4\,y\left(y^2 + 3\alpha y + 1\right), \tag{6}$$

with a (double) pole at zero.

An essential elliptic function $\mathcal{R}_\alpha$ differs from Weierstrass elliptic function $\wp_\alpha$ by an additive constant. Explicitly, if $\wp_\alpha$ is the Weierstrass elliptic function, satisfying the differential equation

$$\begin{gathered} y'^2 = 4\left(y^3 - (3\alpha^2-1)\,y - \alpha(1-2\alpha^2)\right) = 4(y-\alpha)\,(y-(\alpha-\beta))\,(y-(\alpha-1/\beta)), \\ \beta := (3\alpha+d)/2,\ \ d^2 := 9\alpha^2 - 4, \end{gathered} \tag{7}$$

then

$$\mathcal{R}_\alpha = \wp_\alpha - \alpha.$$

In particular, the essential elliptic function $\mathcal{R}_\alpha$ coincides with the Weierstrass function $\wp_\alpha$ when $\alpha = 0$. Incidentally, the discriminant of either the cubic appearing on the right hand side of equation (6) or the cubic in (7), which we shall call *the discriminant associated with* α, is $d^2 = (\beta - 1/\beta)^2 = 9\alpha^2 - 4$, so it does not vanish since, by assumption, $\alpha \neq \pm 2/3$.

Each Weierstrass elliptic function $\wp$, satisfying a differential equation (2), represents a family of Weierstrass elliptic functions $\mathcal{F}_\wp$ (3). Fix such a function $\wp$, representing such a family $\mathcal{F}_\wp$, so as to assume the pair (h_2, h_3) and its corresponding discriminant Δ being fixed. Let τ be the unique point in $\mathcal{D}$ whose image $j = j(\tau)$, given by (1), matches the j-invariant of this family. We shall associate six essential elliptic functions with τ, and thus with $\mathcal{F}_\wp$, as long as τ is neither a cube root of 1 nor a square root of -1.

If c is a root of the hexic

$$c^6 - \frac{3h_2}{\Delta} c^2 + \frac{1}{\Delta} \tag{8}$$

then the differential equation (4), satisfied by $\wp^c$ for that particular choice of c, acquires the form of the differential equation (7) satisfied by $\wp_\alpha$, where α is determined by the choice of c. The assumption $\tau \neq i$ implies that $h_3 \neq 0$ and the hexic (8), then being separable, possesses six distinct roots, which we might enlist in three pairs $(c_1, -c_1)$, $(c_2, -c_2)$ and $(c_3, -c_3)$, where

$$c_1^2 = \frac{1}{(e_1 - e_2)(e_1 - e_3)}, \; c_2^2 = \frac{1}{(e_2 - e_3)(e_2 - e_1)}, \; c_3^2 = \frac{1}{(e_3 - e_1)(e_3 - e_2)}.$$

For each root pair $(c_n, -c_n)$, $n = 1, 2, 3$, we obtain a corresponding factorization of the cubic on the right hand side of (4), with c replaced by c_n and $-c_n$, respectively

$$\begin{cases} y^3 - c_n^2 h_2 y - c_n^3 h_3 = (y - \alpha_n)\,(y - (\alpha_n - \beta_n))\,(y - (\alpha_n - 1/\beta_n)) \\ \\ y^3 - c_n^2 h_2 y + c_n^3 h_3 = (y + \alpha_n)\,(y + (\alpha_n - \beta_n))\,(y + (\alpha_n - 1/\beta_n)) \end{cases}$$

Here, the additional assumption $\tau \neq e^{2\pi i/3}$ guarantees that the three values α_1, α_2 and α_3 are pairwise distinct, whereas we have already ensured, with the assumption $\tau \neq i$, that none of the them vanishes. Thus, six distinct Weierstrass functions $\wp_{\alpha_n}$, $\wp_{-\alpha_n}$, $n = 1, 2, 3$, are obtained, and we may, as promised, associate six essential elliptic functions $\mathcal{R}_{\alpha_n}$, $\mathcal{R}_{-\alpha_n}$, $n = 1, 2, 3$, with $\mathcal{F}_\wp$. Plainly, for each n, $n = 1, 2, 3$, we have

$$\mathcal{R}_{\alpha_n} = \wp_{\alpha_n} - \alpha_n, \; \mathcal{R}_{-\alpha_n} = \wp_{-\alpha_n} + \alpha_n, \quad \alpha_n = c_n e_n,$$

and for each n, $n = 1, 2, 3$, the discriminant associated with α_n is

$$d_n^2 = 9\alpha_n^2 - 4 = c_n^6 \Delta.$$

The three pairs $(d_1, -d_1)$, $(d_2, -d_2)$ and $(d_3, -d_3)$ can be viewed as the roots of the hexic

$$4\left(d^2 + 1\right)^3 - 27jd^2 = 4\left(d^6 + 3d^4 + 3\,(1 - 9j/4)\,d^2 + 1\right), \tag{9}$$

which is separable aside from the two special cases corresponding to $\tau = i$ and $\tau = e^{2\pi i/3}$. We now consider these two cases.

If $\tau = i$ then $h_3 = 0$, $j = 1$ and the hexic (9) factors to

$$4\left(d^2+4\right)\left(d^2-1/2\right)^2 .$$

If $\tau = e^{2\pi i/3}$ then $h_2 = 0$, $j = 0$ and the hexic (9) factors to

$$4\left(d^2+1\right)^3 .$$

Instead of writing the hexic (9) we could have written a hexic which roots are the three pairs $(\alpha_1,\, -\alpha_1)$, $(\alpha_2,\, -\alpha_2)$ and $(\alpha_3,\, -\alpha_3)$. This is the hexic

$$4\left(3\alpha^2-1\right)^3 - j\left(9\alpha^2-4\right). \tag{10}$$

Only three distinct roots and three essential elliptic functions correspond to $\tau = i$. These are

$$\left\{\mathcal{R}_\alpha :\ \alpha = 0,\ \pm\sqrt{2}/2\right\}.$$

Only two distinct roots and two essential elliptic functions correspond to $\tau = e^{2\pi i/3}$. These are

$$\left\{\mathcal{R}_\alpha :\ \alpha = \pm\sqrt{3}/3\right\}.$$

The hexic polynomials (8), (9) and (10), when viewed as cubic polynomials of c^2, d^2 and α^2, respectively, correspond to one and the same point in $\mathcal{D}$. In other words, theses three cubics possess the same j-invariant. The latter is obtained from the j-invariant of the original cubic (2) via the transformation

$$j \mapsto \frac{j}{j-1}$$

which constitutes a linear fractional transformation of order two, with zero being its only fixed point. Note that the discriminant of the cubic (8) for c^2 does not vanish when $\tau = e^{2\pi i/3}$, although it does for either one of the cubic polynomials (9) for d^2 or (10) for α^2.

We may "restore" the fundamental domain by adding two functions, corresponding to the two excluded values $\pm 2/3$ for α. When $\alpha = \pm 2/3$ equation (6) degenerates to

$$y'^2 = 4\,y\,(y \pm 1)^2 ,$$

and we, thus, regard the functions ctg^2 and cth^2 as the functions corresponding to $\alpha = 2/3$ and $\alpha = -2/3$, respectively, where

$$\mathrm{ctg}^2(x) = -\left(\frac{e^{2ix}+1}{e^{2ix}-1}\right)^2, \quad \mathrm{cth}^2(x) = \left(\frac{e^{2x}+1}{e^{2x}-1}\right)^2 .$$

3. The Galois alternative elliptic function

Define an *alternative elliptic function* $\mathcal{S}_\beta$ as the vanishing at zero solution of the differential equation

$$y'^2 = \left(1-\beta y^2\right)\left(1-y^2/\beta\right),$$

whose leading Maclaurin series coefficient is 1.

Set the elliptic modulus k equal to β. The alternative elliptic function is then a representative of the family $\mathcal{F}_{\mathrm{sn}}$ introduced in (5), namely

$$\mathcal{S}_\beta = H\left(\sqrt{\beta}\right)\mathrm{sn},$$

and its square is the reciprocal of an essential elliptic function, namely

$$\mathcal{S}_\beta^2 = 1/\mathcal{R}_{-\alpha}. \tag{11}$$

The degenerate elliptic function $\mathcal{S}_{-1}$ and $\mathcal{S}_1$ correspond to $\alpha = 2/3$ and $\alpha = -2/3$, respectively

$$\mathcal{S}_{-1}(x) := \tan(x) = \frac{e^{2ix}-1}{i\left(e^{2ix}+1\right)}, \quad \mathcal{S}_1(x) := \mathrm{tgh}(x) = \frac{e^{2x}-1}{e^{2x}+1}.$$

4. Essential relations amongst Galois elliptic functions

With

$$k^2 = \frac{3\alpha-2}{3\alpha+2} = \left(\frac{\beta-1}{\beta+1}\right)^2,$$

the function

$$\mathcal{S} := H\left(\sqrt{\frac{1}{d}}\right)\mathcal{S}_k = H\left(\sqrt{\frac{k}{d}}\right)\mathrm{sn}$$

satisfies the differential equation

$$y'^2 = \left(1-(3\alpha+2)\,y^2\right)\left(1-(3\alpha-2)\,y^2\right),$$

and, moreover, the identity

$$\mathcal{S}^{-2} = \mathcal{R}_\alpha + 3\alpha + \mathcal{R}_\alpha^{-1} = \mathcal{W}_\alpha + 2\alpha, \tag{12}$$

where the sum

$$\mathcal{W}_\alpha := \mathcal{R}_\alpha + \alpha + \mathcal{R}_\alpha^{-1}$$

is viewed as a Weierstrass function, satisfying the equation

$$y'^2 = 4\,(y+2\alpha)\,(y-\alpha+2)\,(y-\alpha-2),$$

holds. Thus, identity (12) establishes a well-known (yet, perhaps, somewhat disguised) link between Weierstrass and Jacobi elliptic functions. The function $\mathcal{W}_\alpha$ represents a family $\mathcal{F}_{\mathcal{W}_\alpha} = \{\mathcal{W}_\alpha^c : \ c \in \mathbb{C}^\times\}$, corresponding to a point $\tau \in \mathcal{D}$. We proceed to determining a particular representative $\mathcal{W}_\alpha^c$ differing from an essential elliptic function, associated with τ, by an additive constant. Evidently, for $c = 1/d = 1/\sqrt{9\alpha^2-4}$,

$$\mathcal{W}_\alpha^c + 2c\alpha = \mathcal{R}_{-2c\alpha}$$

is then satisfied by that, sought for, representative $\mathcal{W}_\alpha^c$, whence

$$\mathcal{W}_\alpha + 2\alpha = \mathcal{R}_{-2c\alpha}^d,$$

and

$$\mathcal{S}^2 = 1/\mathcal{R}_{-2c\alpha}^d,$$

reestablishing that

$$\mathcal{S}_k^2 = 1/\mathcal{R}_{-2c\alpha},$$

in agreement with formula (11).
Applying the identity

$$\frac{1}{4}\left(\frac{\mathcal{R}'_\alpha}{\mathcal{R}_\alpha}\right)^2 = \mathcal{R}_\alpha + 3\alpha + \mathcal{R}_\alpha^{-1},$$

to formula (12), we arrive at the equalities

$$\mathcal{S}^{-2} = \frac{1}{4}\left(\frac{\mathcal{R}'_\alpha}{\mathcal{R}_\alpha}\right)^2 = \left(\frac{\mathcal{S}'_{-\beta}}{\mathcal{S}_{-\beta}}\right)^2,$$

implying the relation

$$\int_{x_0}^{x} \mathcal{S}^{-1}(x)\,dx = \ln\left(\mathcal{S}_{-\beta}\left(x\right)\right),$$

where $2x_0$ is the root of $\mathcal{R}_\alpha$ closest to zero. Globally, of course, the latter equality holds true "modulo $2\pi i$".

Uncited relevant references

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