Numerical Symbolic Dynamics: Complexity of Finite Sequences

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Numerical Symbolic Dynamics: Studies of the Invariant Components

Tisagh Chase, [1] *Aleksandr Mylläri,* [1] *Tatiana Mylläri,* [1] *Nikolay Vassiliev,* [2]

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Numerical Symbolic Dynamics

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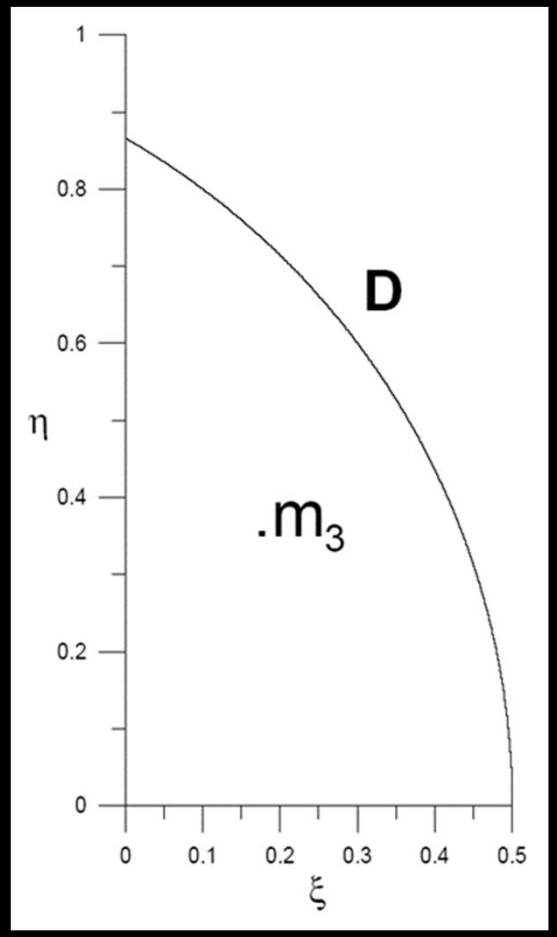
Astronomers have found a distant galaxy with a trio of tightly bound supermassive black holes looming in its central region.

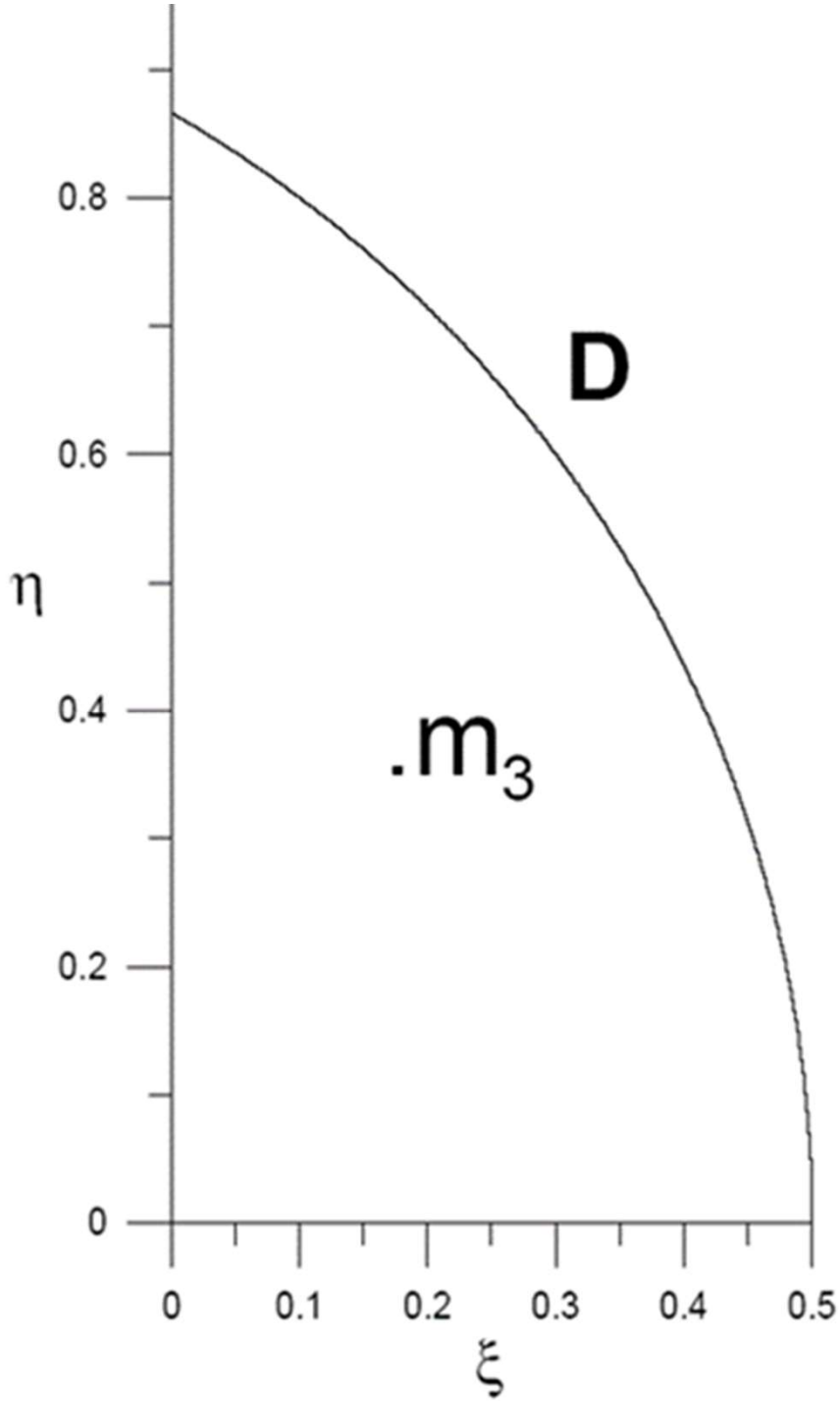
Initial conditions

In the general case:

3 masses of the bodies
+
9 initial coordinates
+
9 initial velocities



Equal-mass free-fall three-body problem:
two (!) coordinates



$$
\begin{aligned}
& z^{120} \\
& + 900\, z^{116} \\
& + 508450\, z^{112} \\
& + 12375000\, z^{110} \\
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& + 24978500000\, z^{106} \\
& + 86356219375\, z^{104} \\
& + 24591849375000\, z^{102} \\
& - 643754458910680\, z^{100} \\
& + 13397352506250000\, z^{98} \\
& - 193920628655076100\, z^{96} \\
& + 5239487723543750000\, z^{94} \\
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& + 102839123473109174525075 00000\, z^{82} \\
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& + 10717989001030340676381749813 6277597607569513276672000000000000\, z^{18} \\
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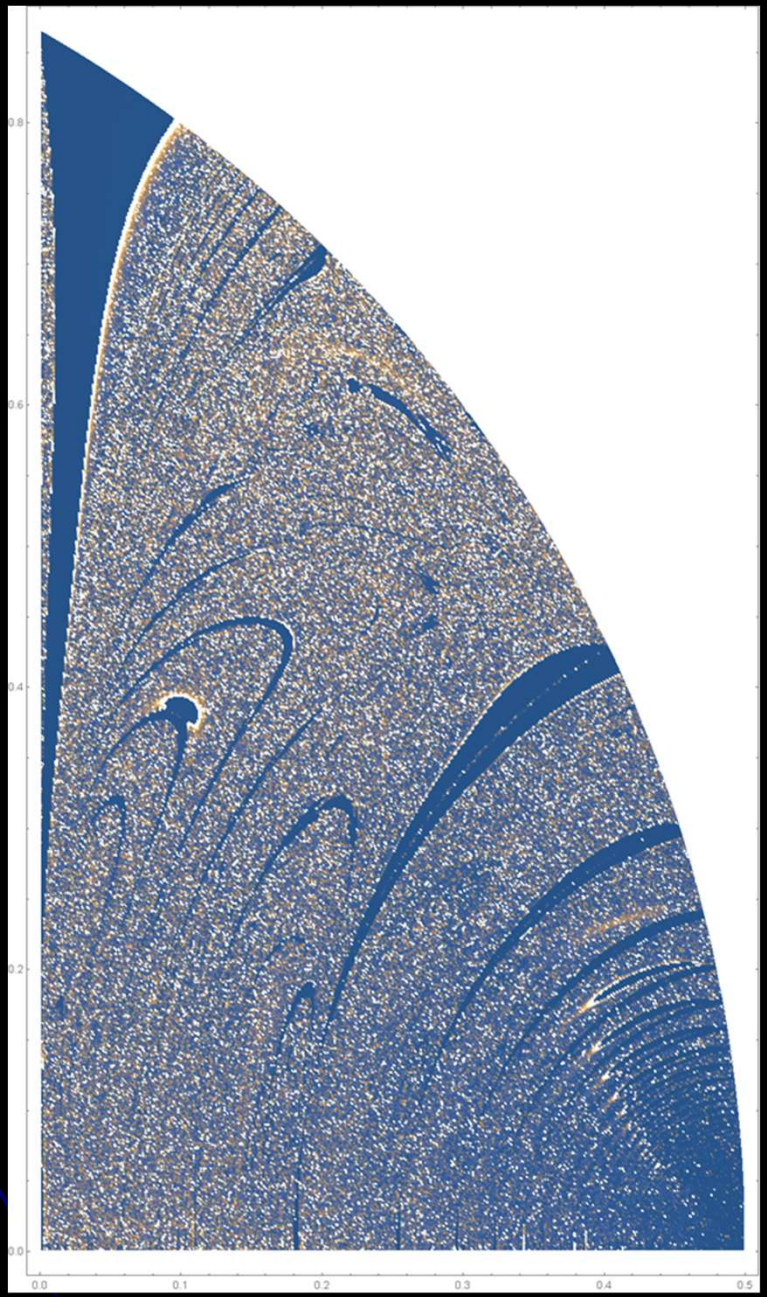
& + 24365639973684917858679310359 2713367811352856903680000000000000\, z^{8} \\
& + 40615589227675119914100324987 8011988348350878208000000000000000\, z^{6} \\
& + 21508468021796121727365079945 7544328788048878080000000000000000000\, z^{4} \\
& + 12368003557410966199224287239 9402881839796300800000000000000000000\, z^{2} \\
& + 5389983298869713157301333780458795052059040000000000000000000000
\end{aligned}
$$

The Burj al Arab Polynomial
(credits: Lorenzo Robbiano)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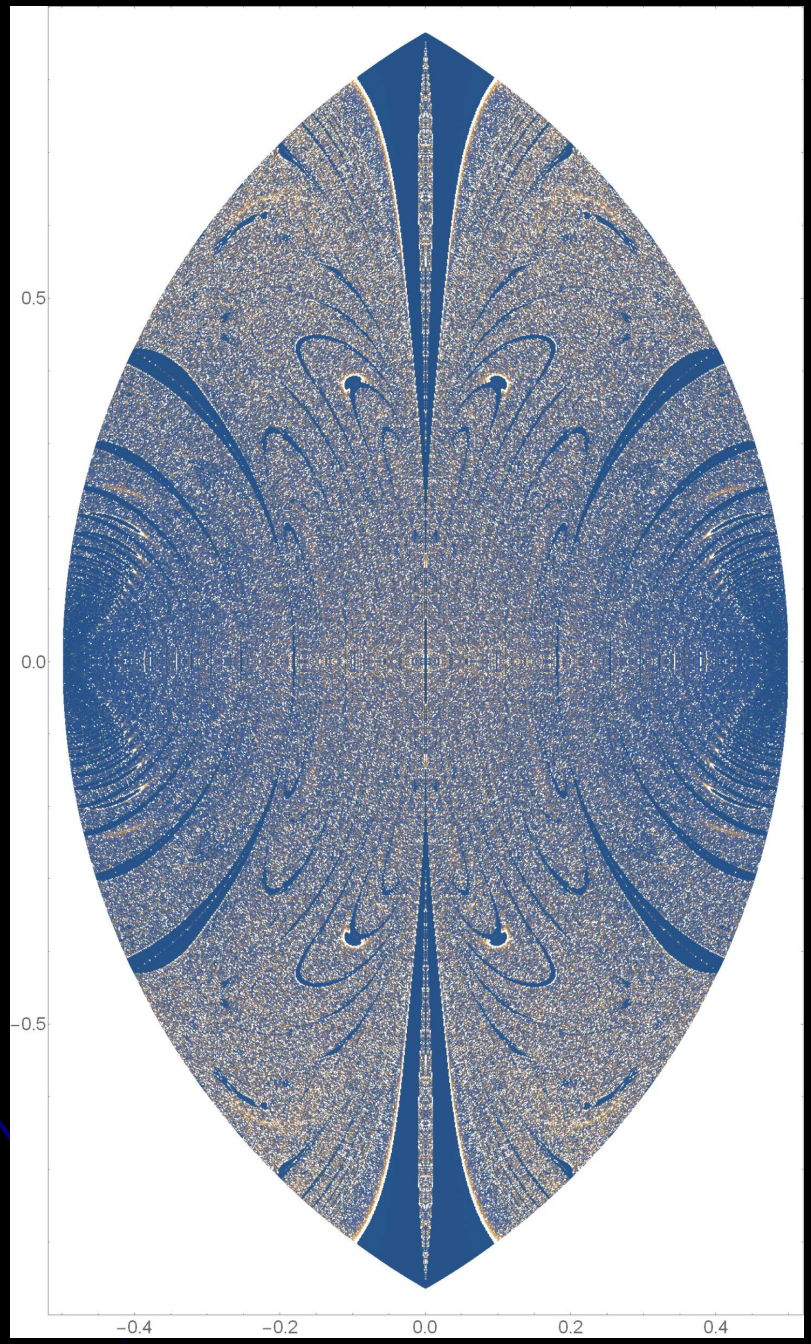
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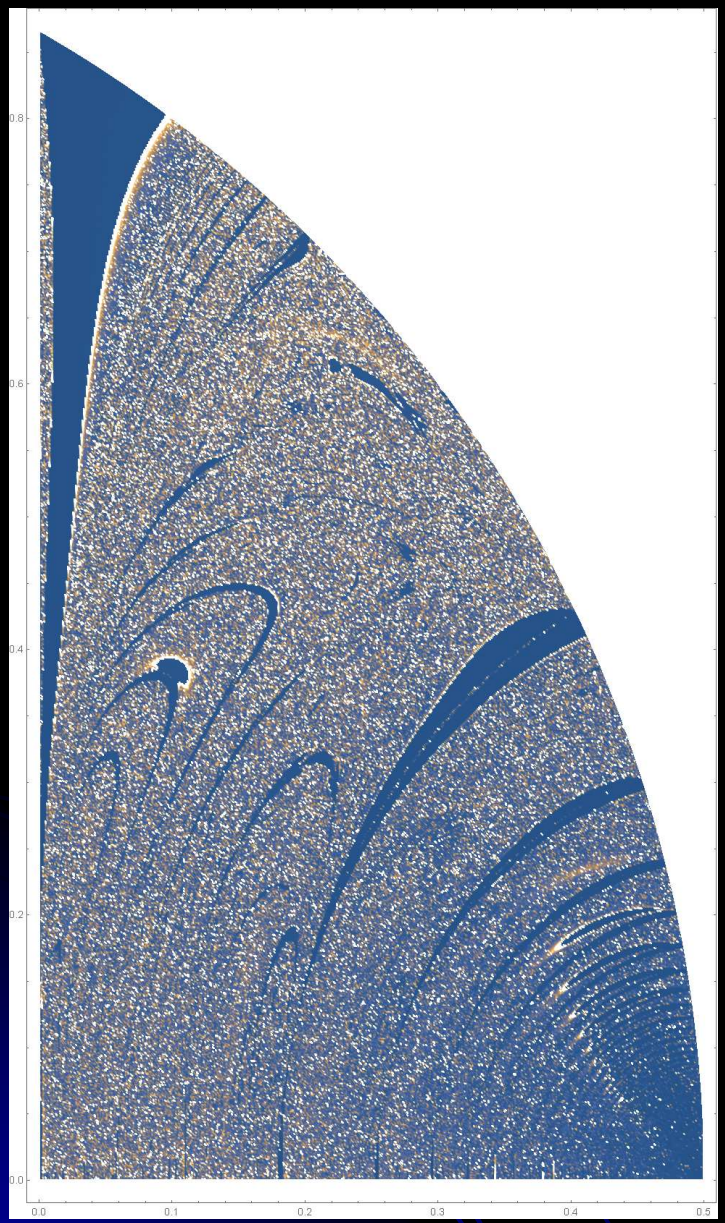
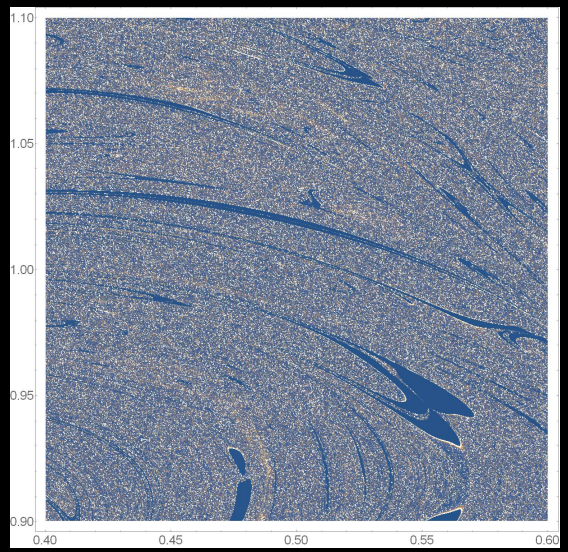
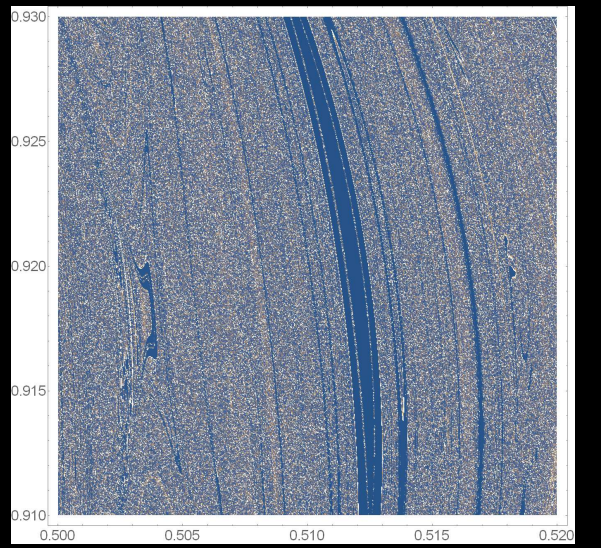
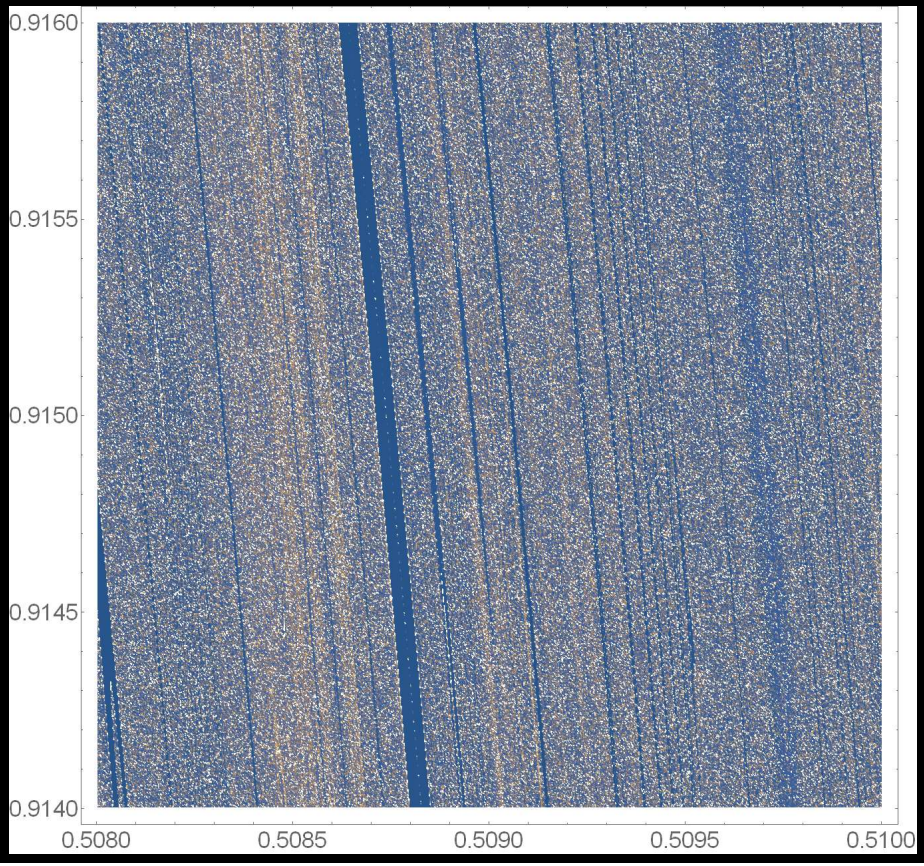
−0.4
−0.2
0.0
0.2
0.4









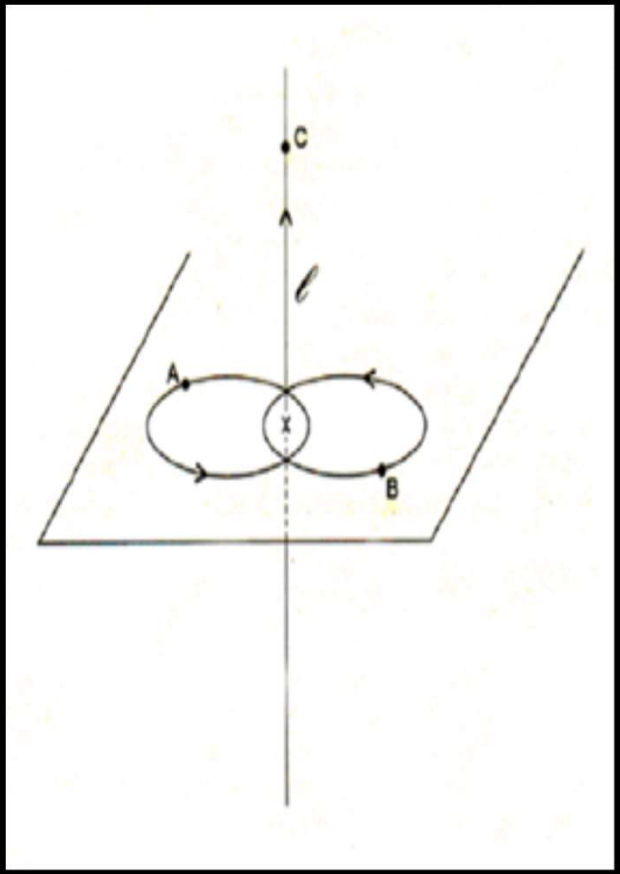
Blue -- short-lived, yellow -- long-lived

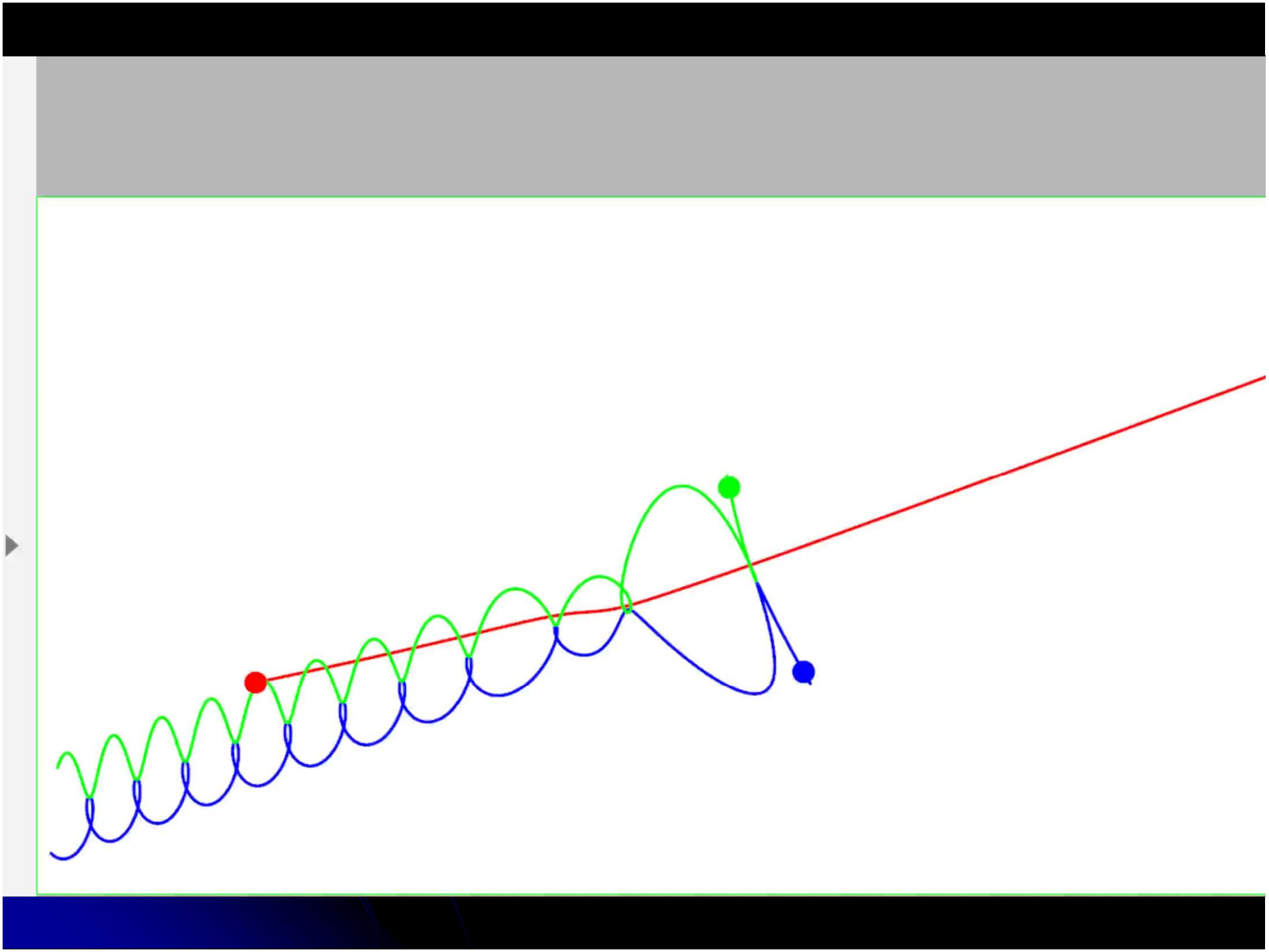
History

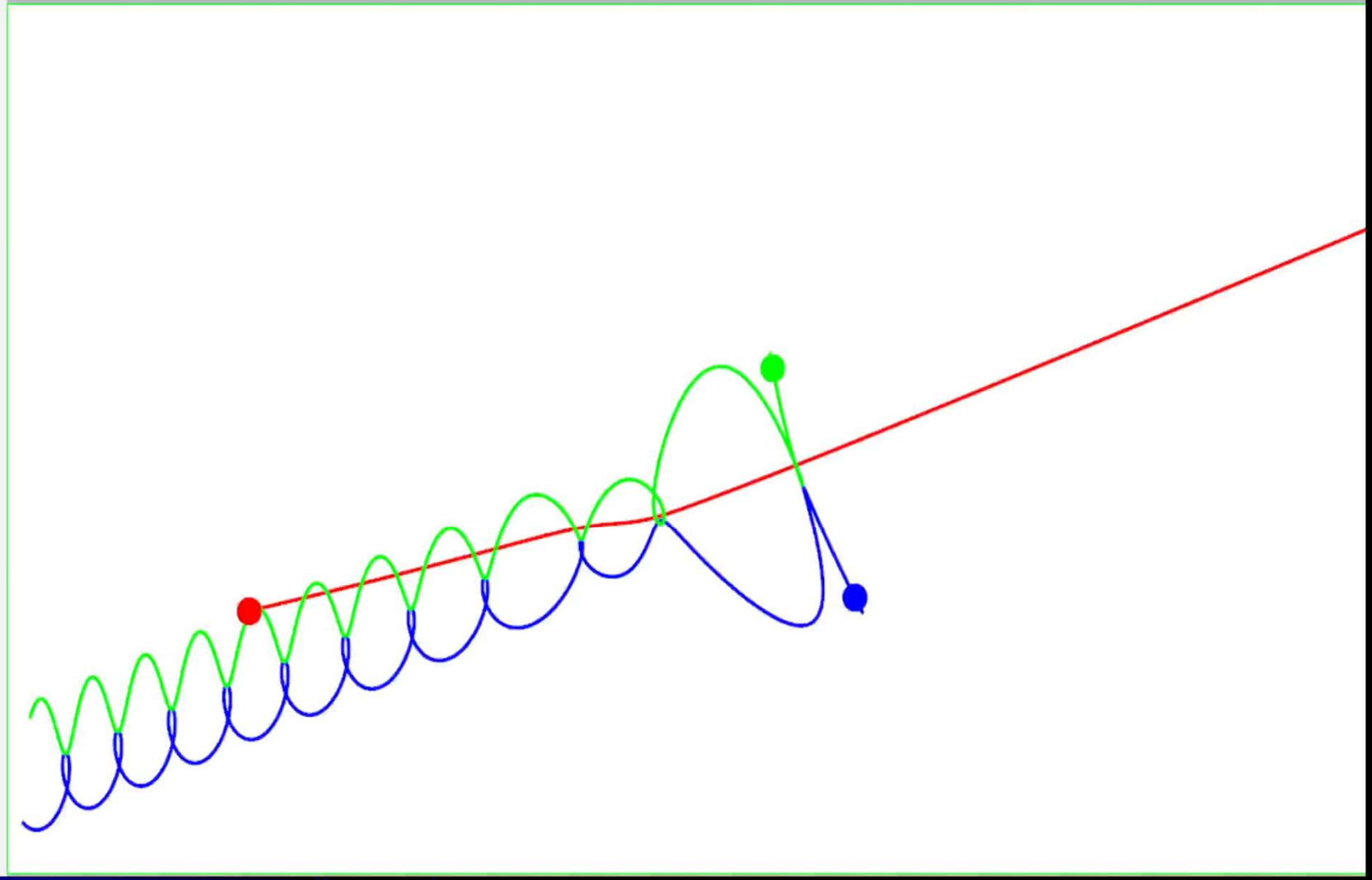
Sitnikov problem
(Alexeev, 1969)
Rectilinear problem
(Tanikawa & Mikkola, 2000)
Isosceles problem
(Zare & Chesley, 1998; Chesley, 1999)
Free-fall equal-mass
three-body problem
(Chernin et al., Mylläri et al., 2004, 2006)

Sitnikov problem

(Alexeev, 1969)



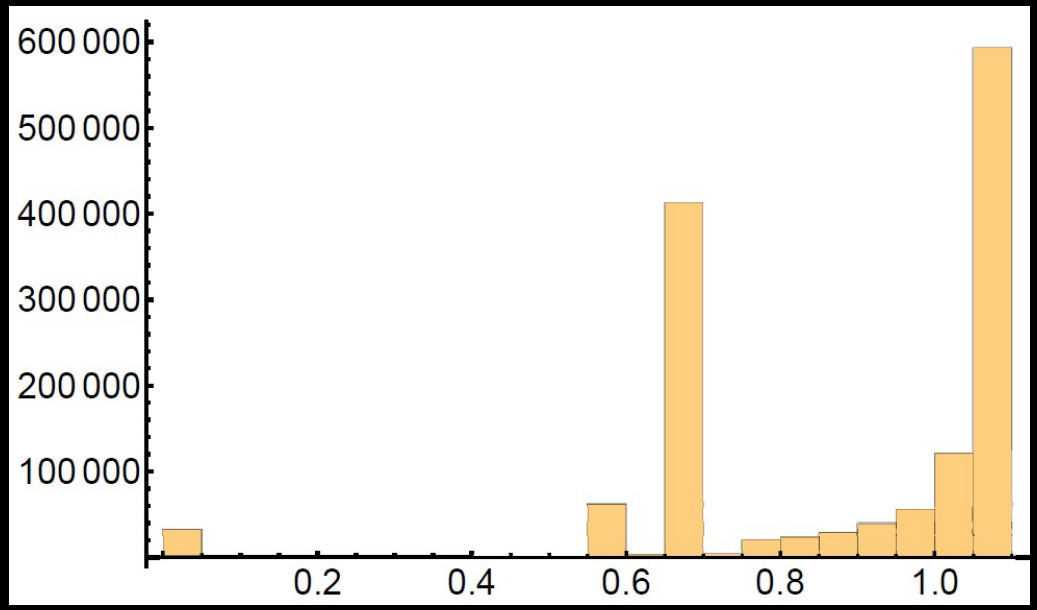




(Shannon) Entropy

$$H_1 = -\Sigma_i p_i \ln p_i$$

p_i – **frequency of symbol *"i"* in the sequence**



- entropy of {1, 2, 1, 2, 1, 2} (or of {1,1,1,1,2,2,2,2}) equals 0.693147

{1,1,1,2,2,2,3,3,3} vs. {1,2,1,3,2,3,2,1,3}

(Shannon) Entropy

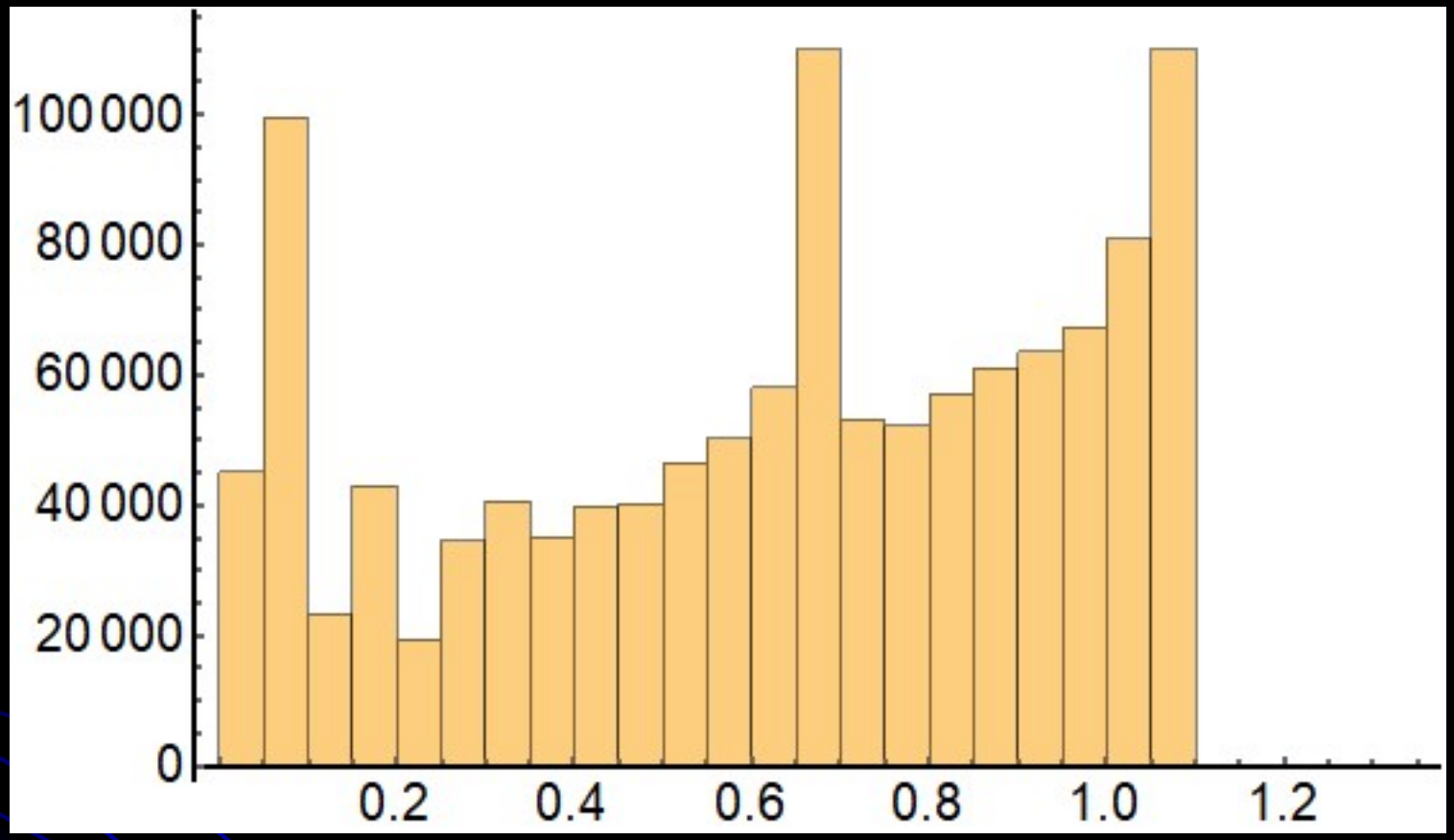
$$H_1 = -\Sigma_i p_i \ln p_i$$

Markov Entropy

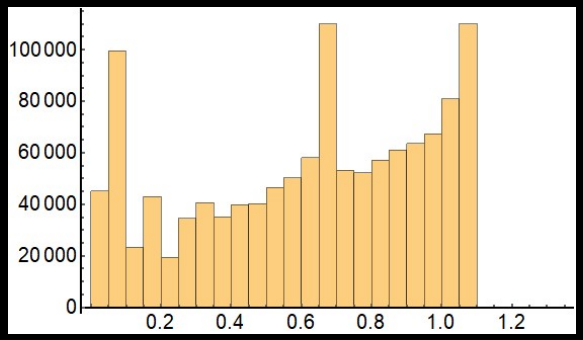
$$H_2 = -\Sigma_i p_i \Sigma_j q_{ij} \ln q_{ij}$$

p_i – **frequency of symbol *"i"* in the sequence;**

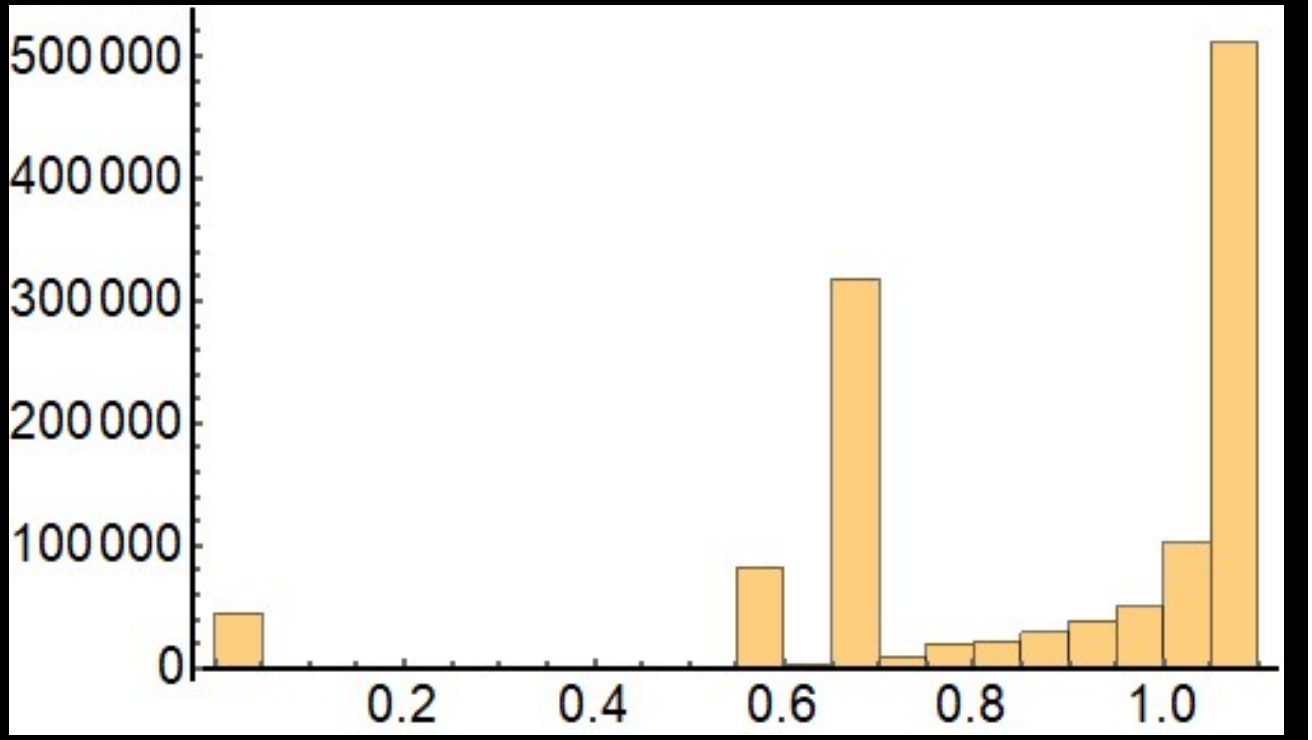
q_{ij} – **frequency of transitions from *"i" to "j"*.**



Binary encounters

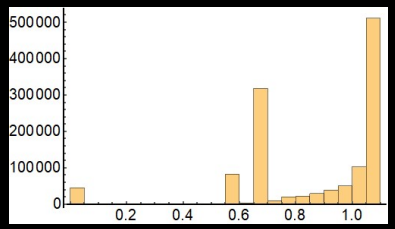


Binary encounters



Entropy[{1, 2, 1, 2}]=0.693147

entropy of {1, 2, 3, 1, 2, 3} (or of {1,1,1,2,2,2,3,3,3}) is 1.09861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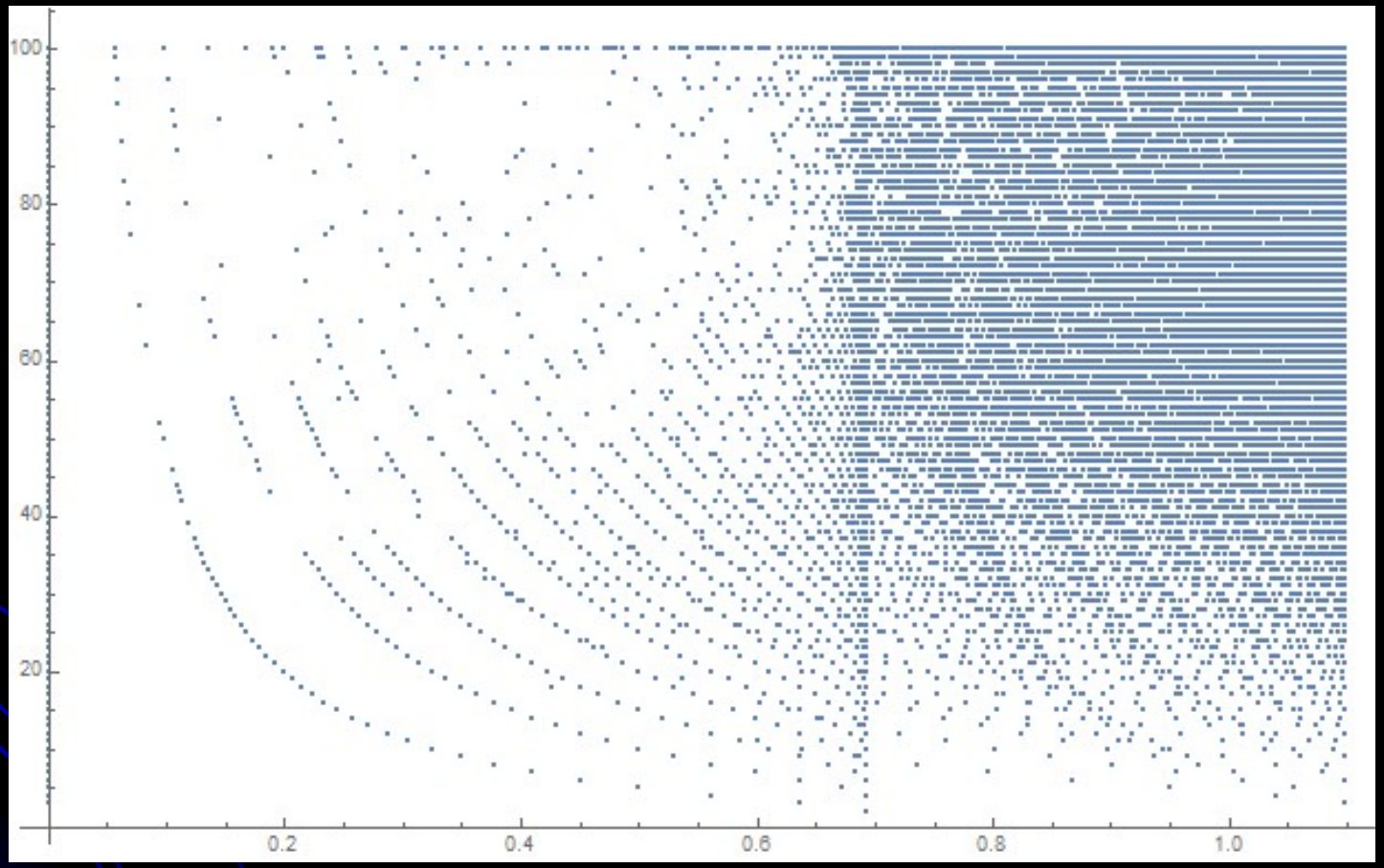
0.8
1.0


100
80
60
40
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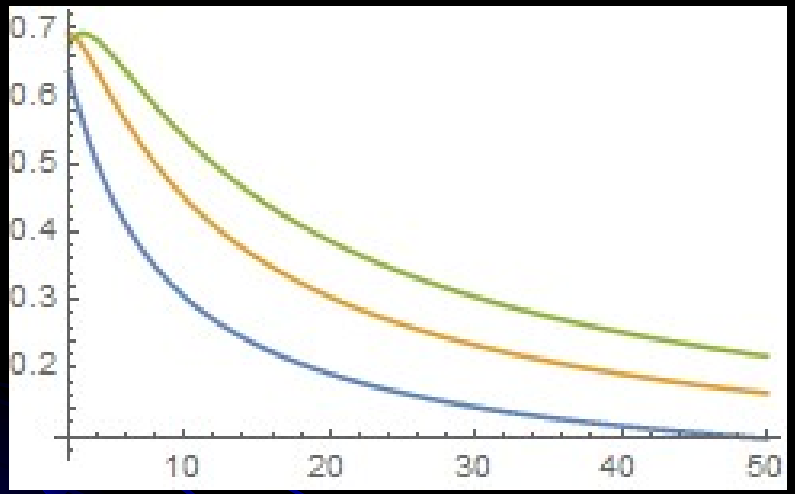
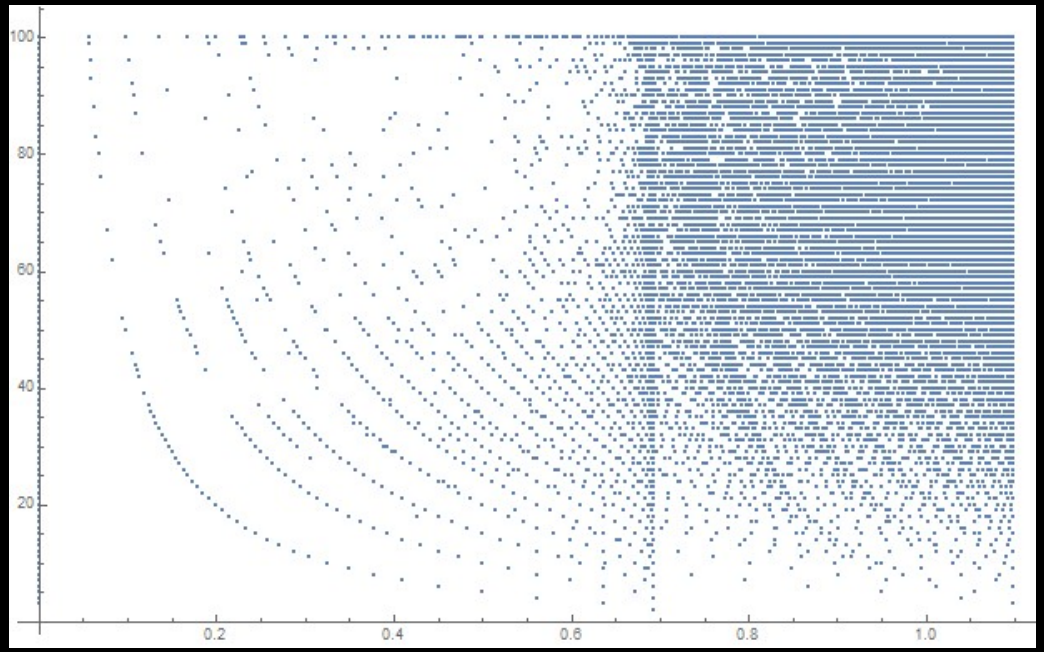
0.2
0.4
0.6
0.8
1.0

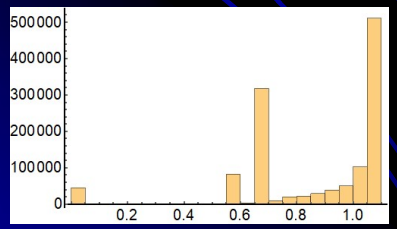
```
f[x_, y_] = -x Log[x / (y + x)] / (y + x) - y Log[y / (y + x)] / (y + x)
```

$$-\frac{x\,\mathrm{Log}\left[\frac{x}{x+y}\right]}{x+y}-\frac{y\,\mathrm{Log}\left[\frac{y}{x+y}\right]}{x+y}$$

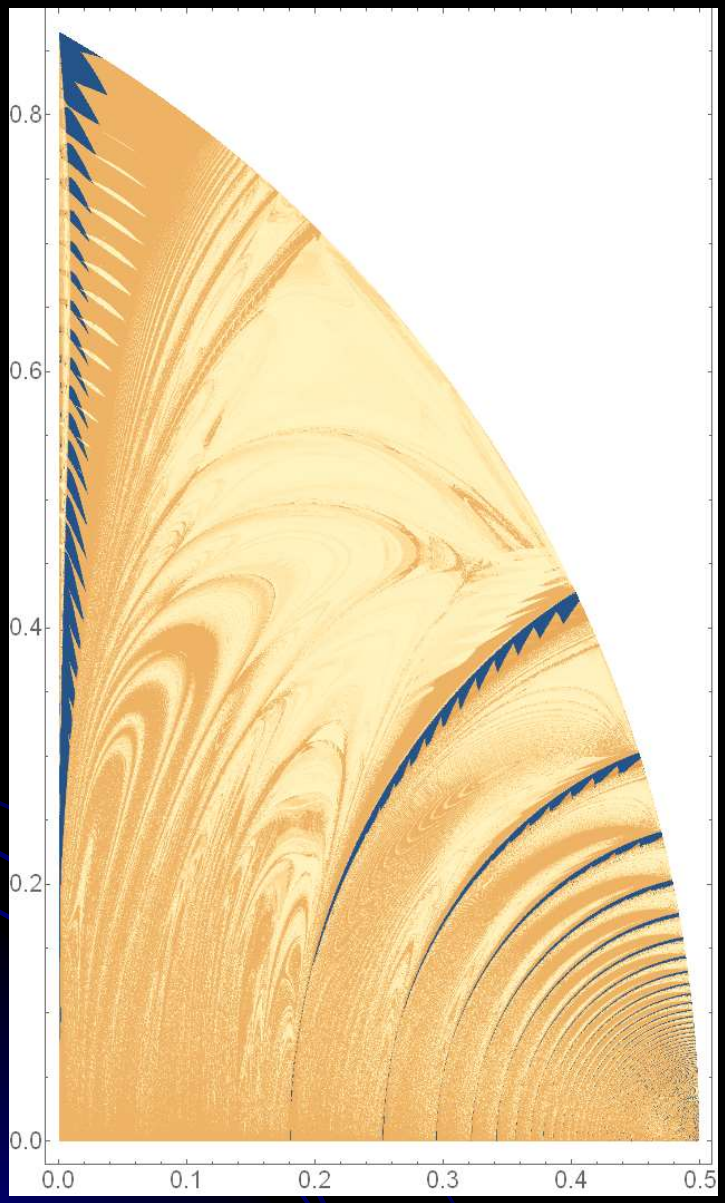
```
Plot[{f[x, 1.], f[x, 2.], f[x, 3.]}, {x, 2., 50.}]
```







Binary encounters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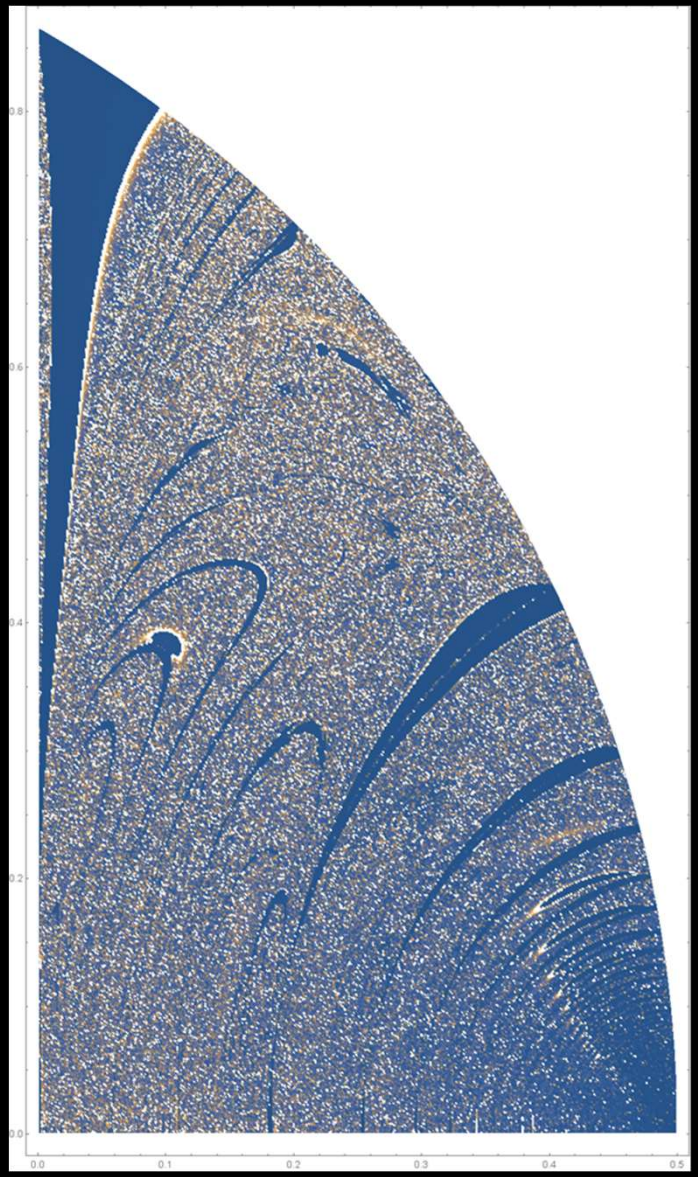
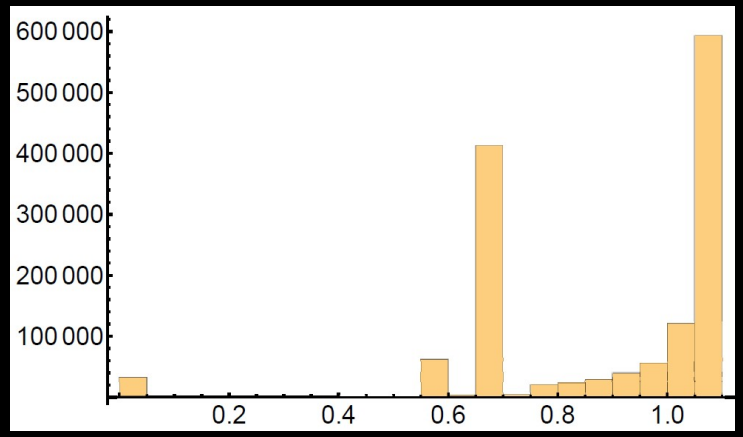
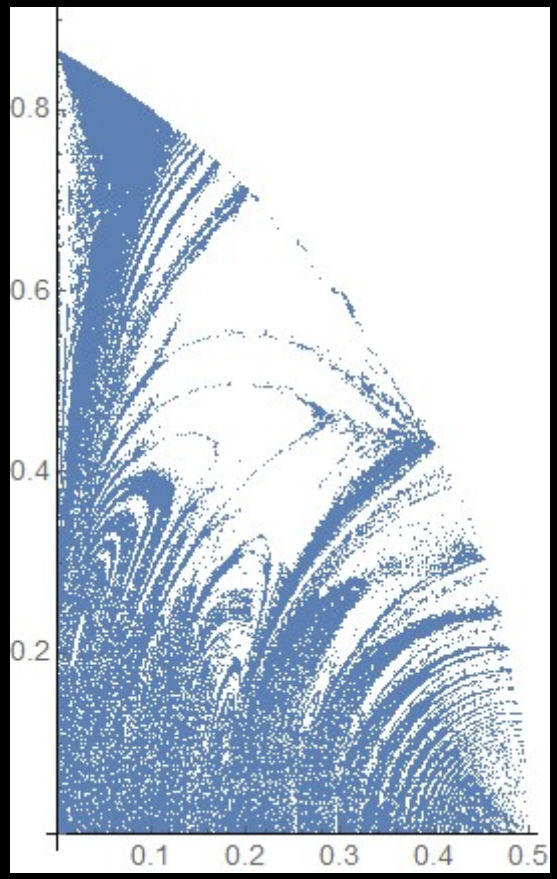
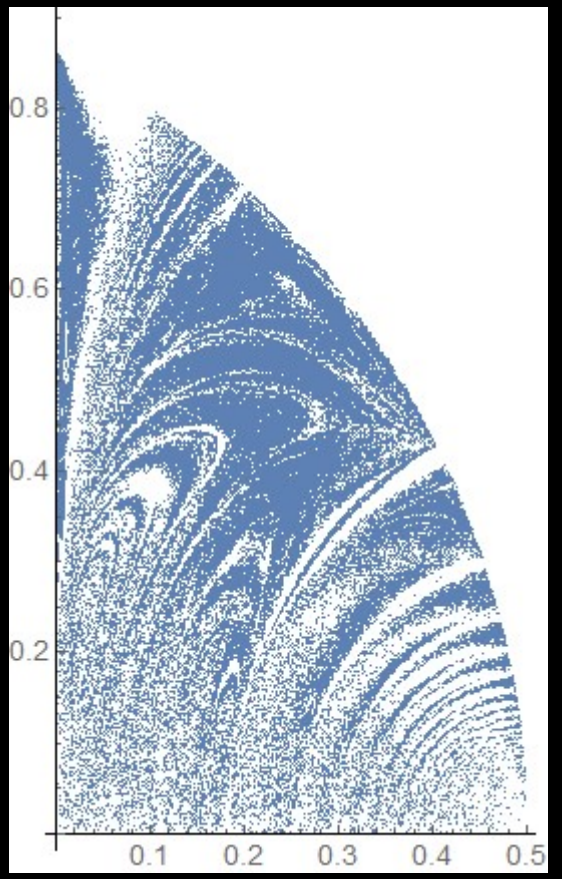
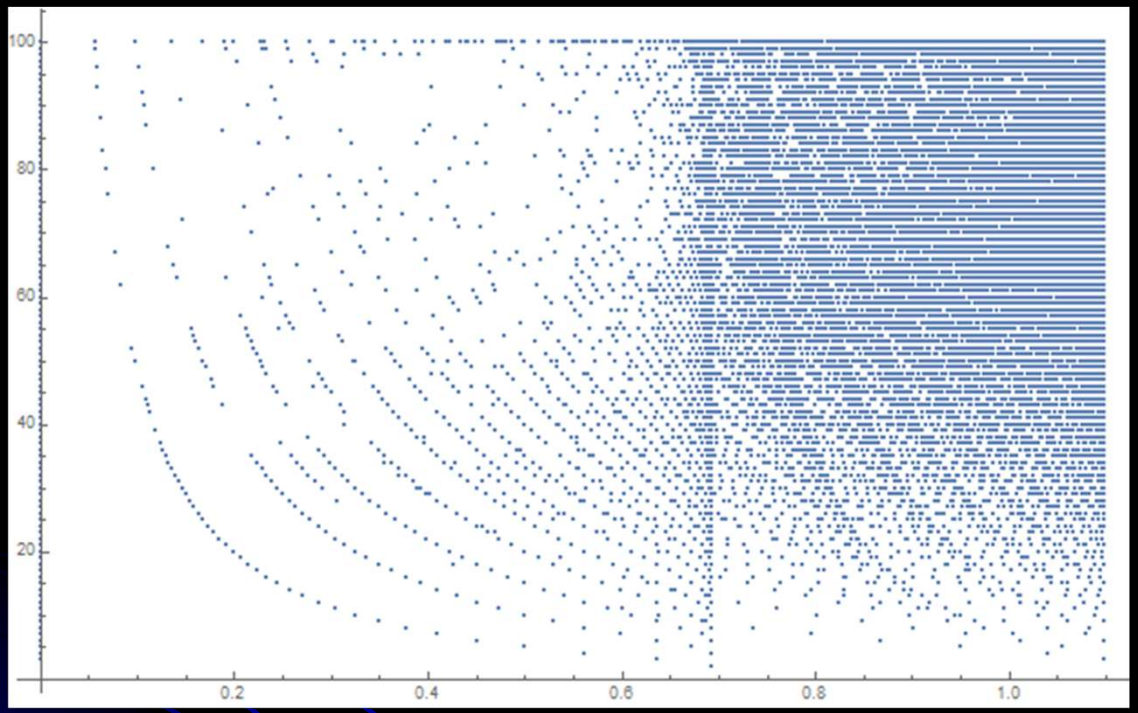








Initial conditions corresponding to the two modes on the histogram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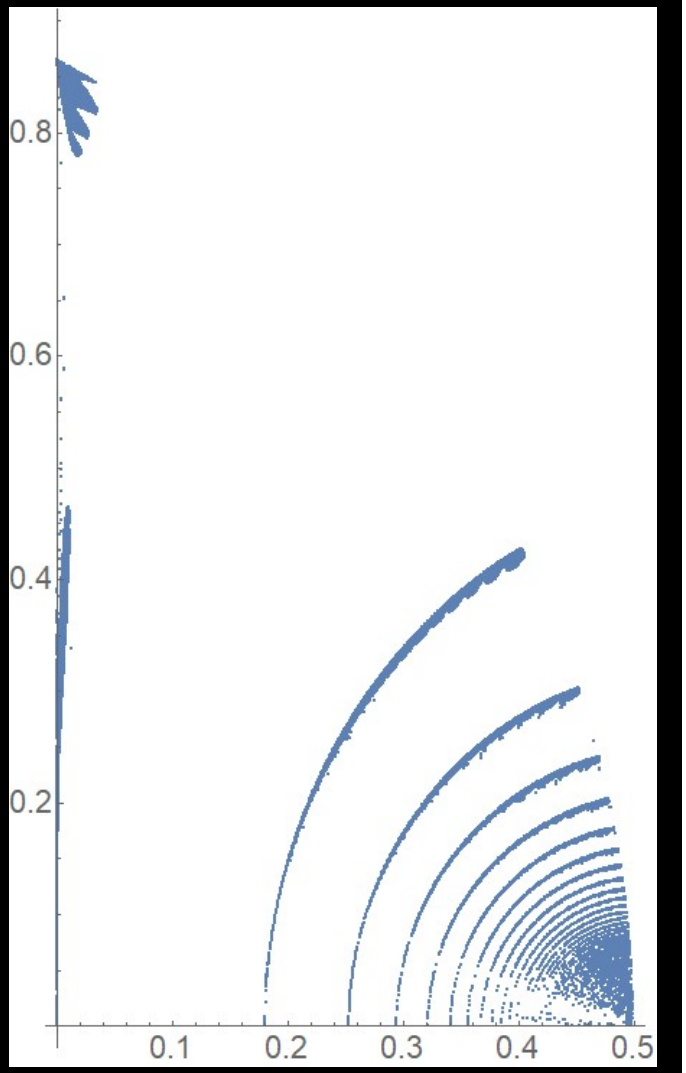
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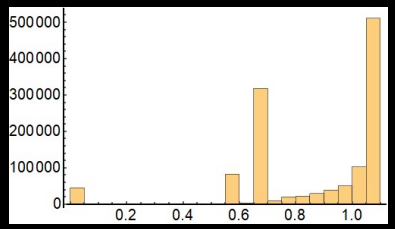
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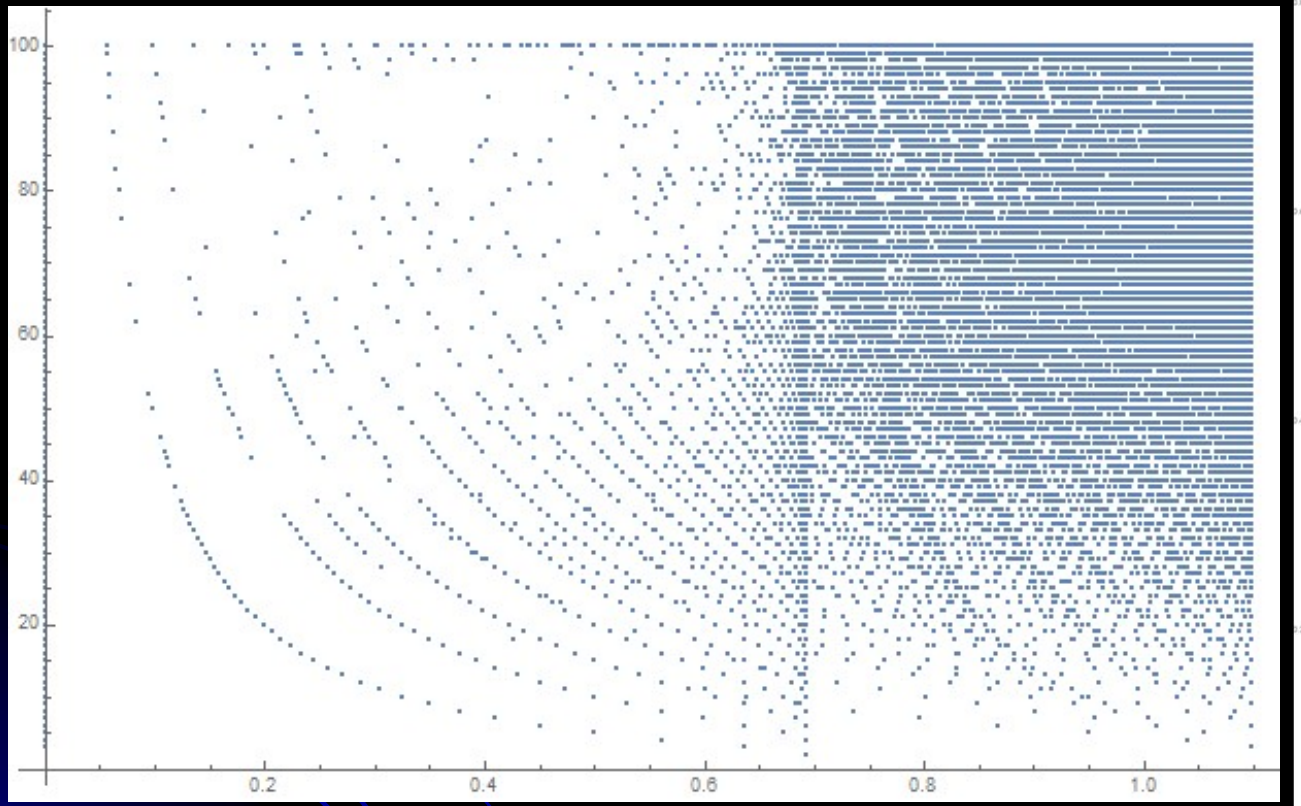
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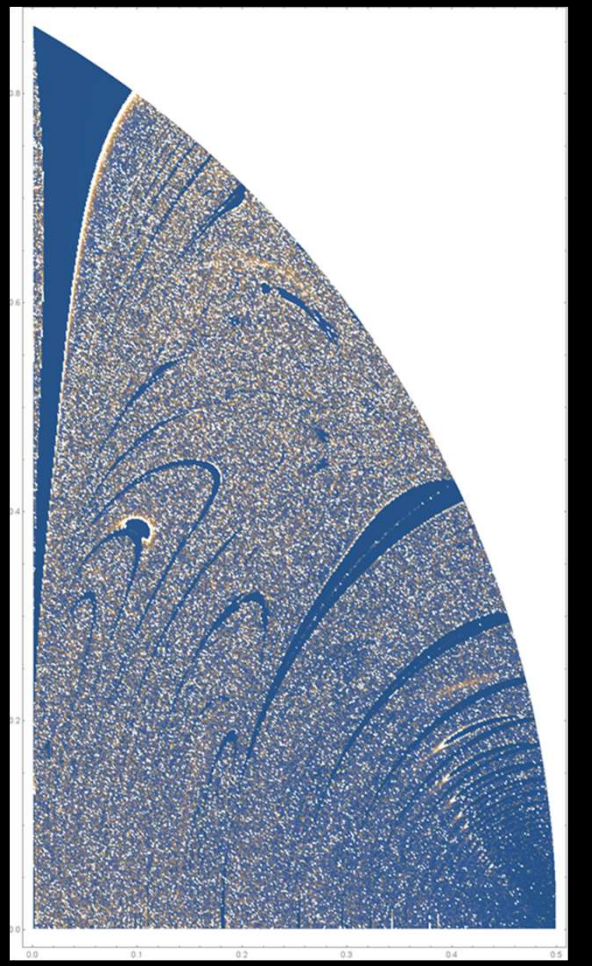
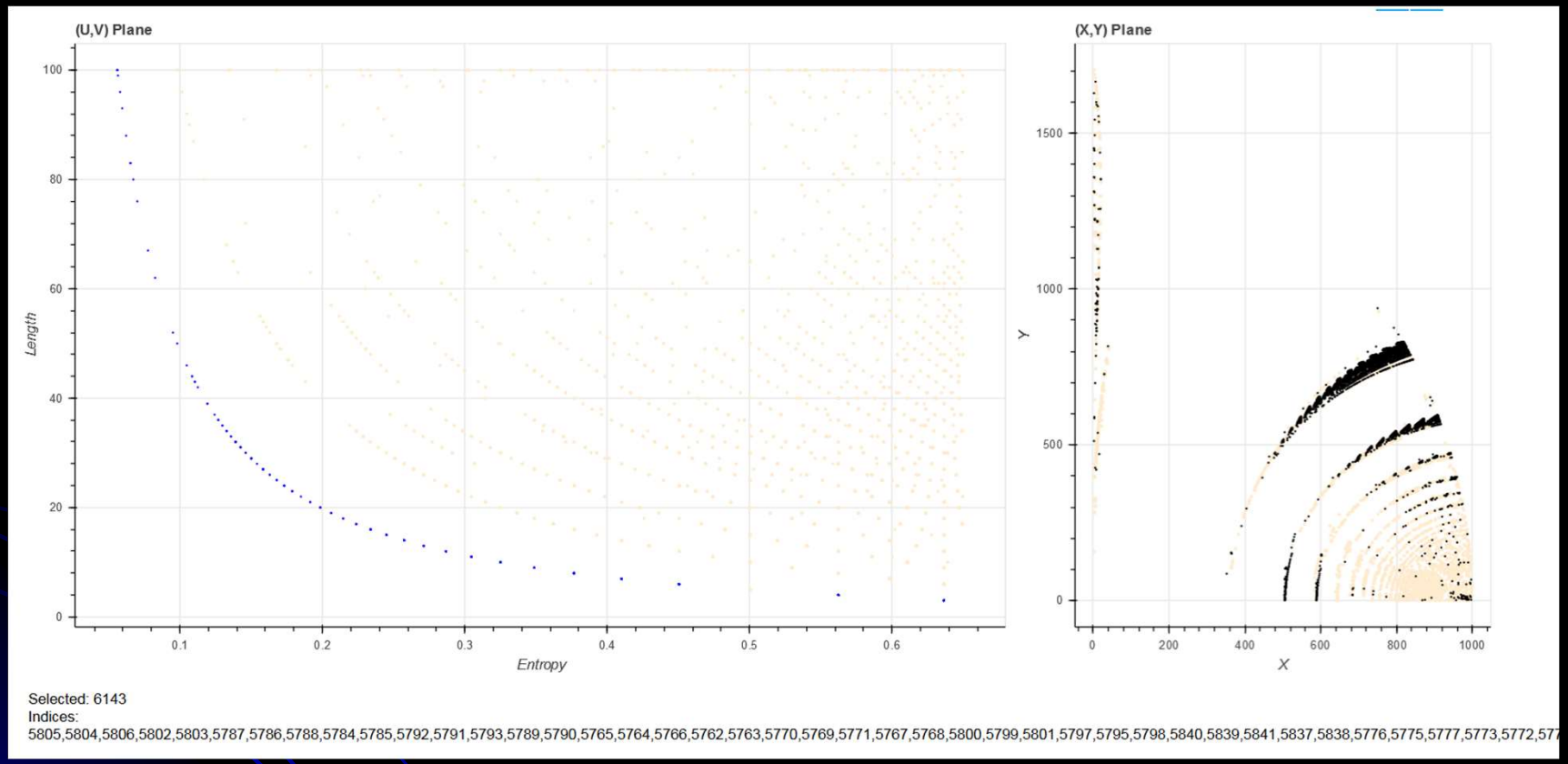
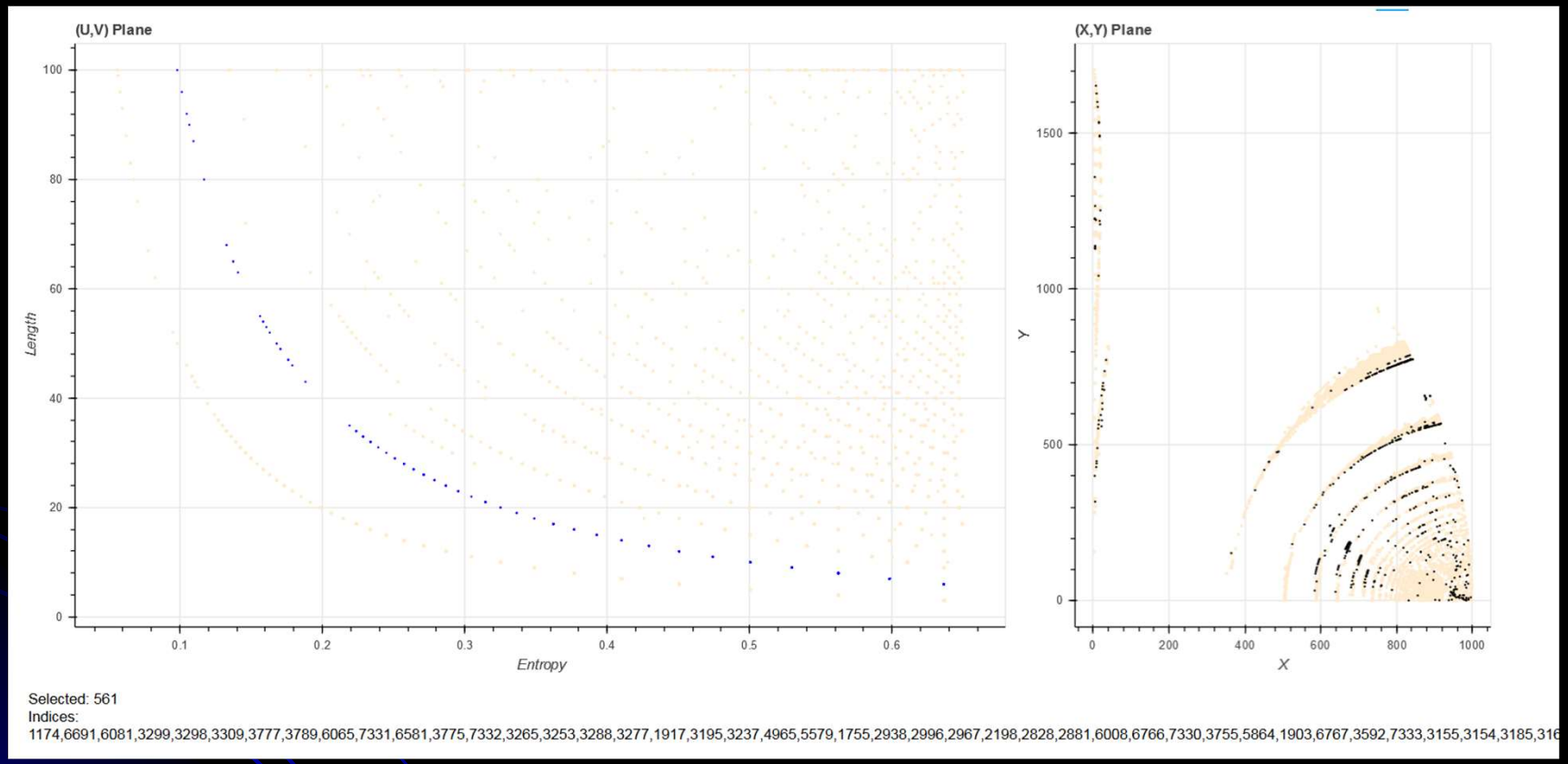
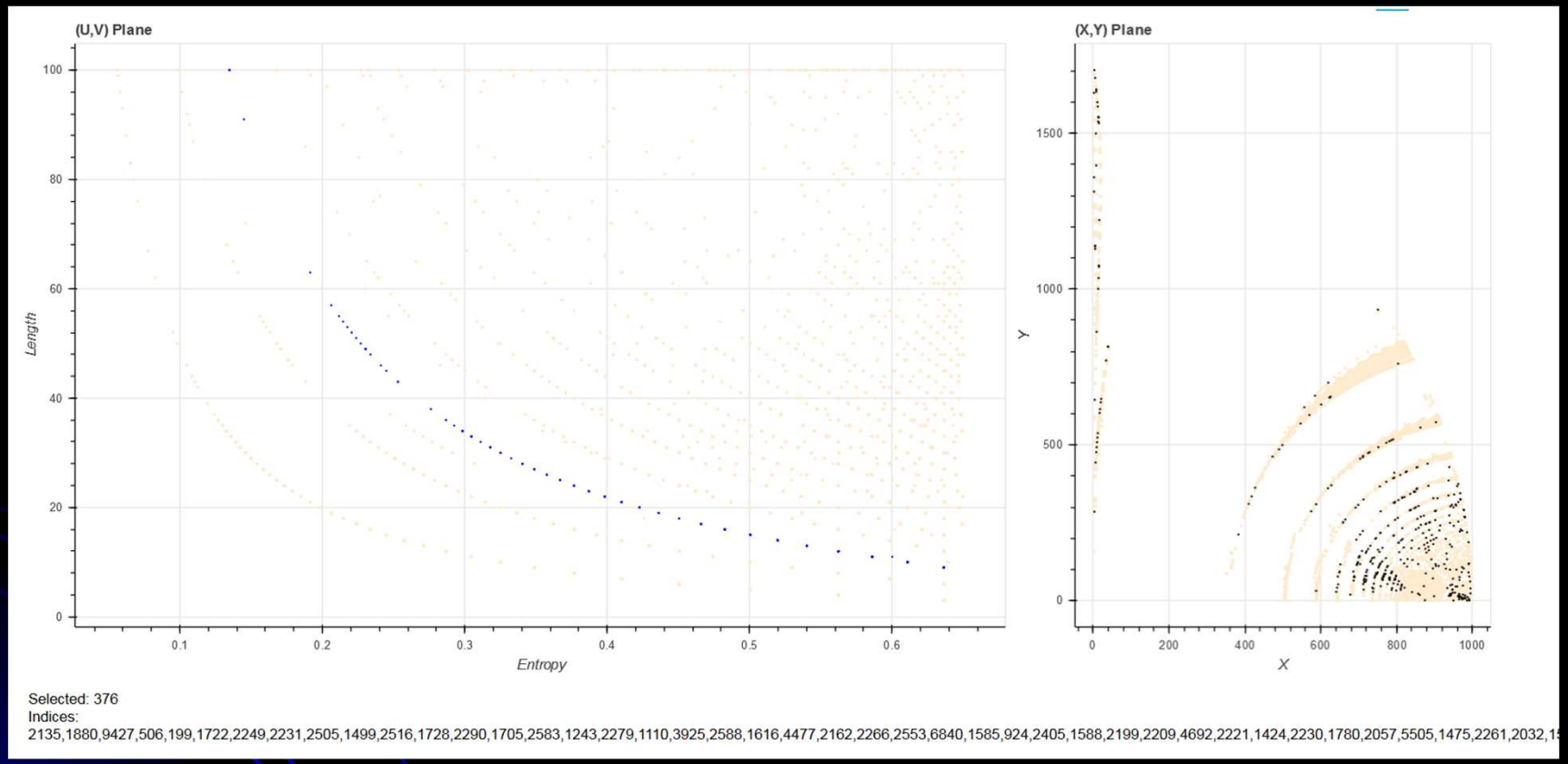




Screenshot 2019-04-12 03.58.45

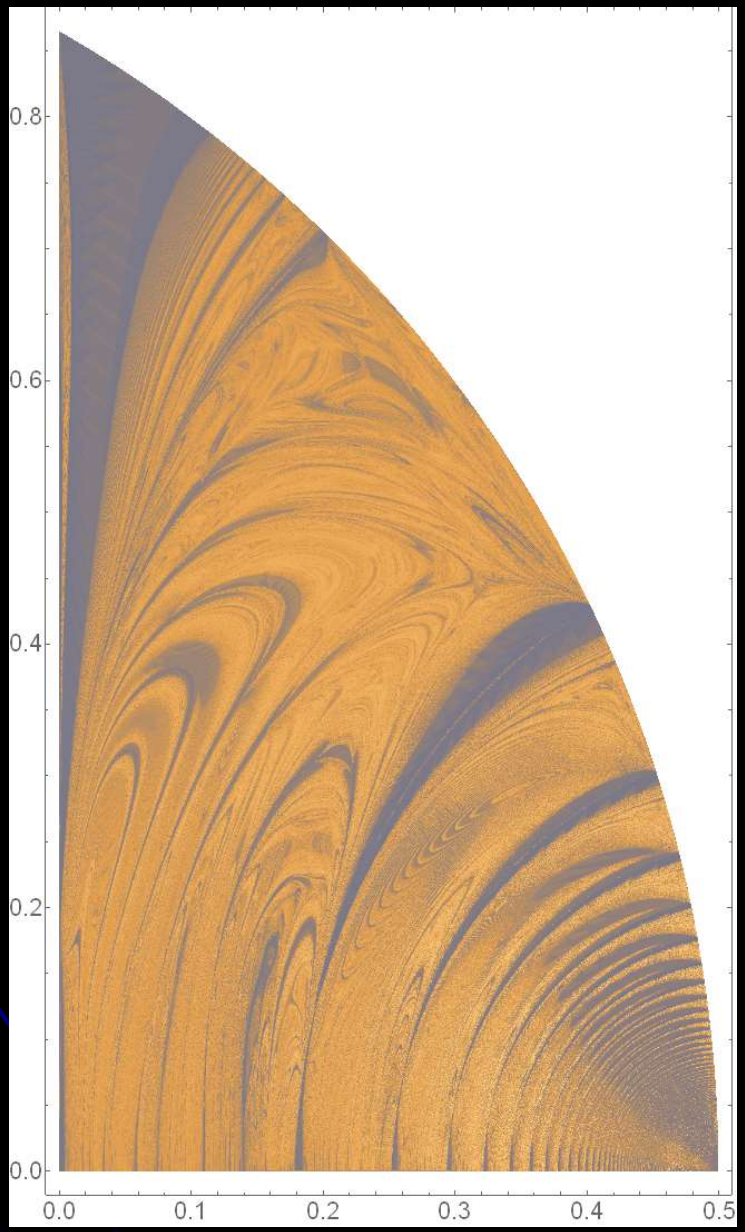


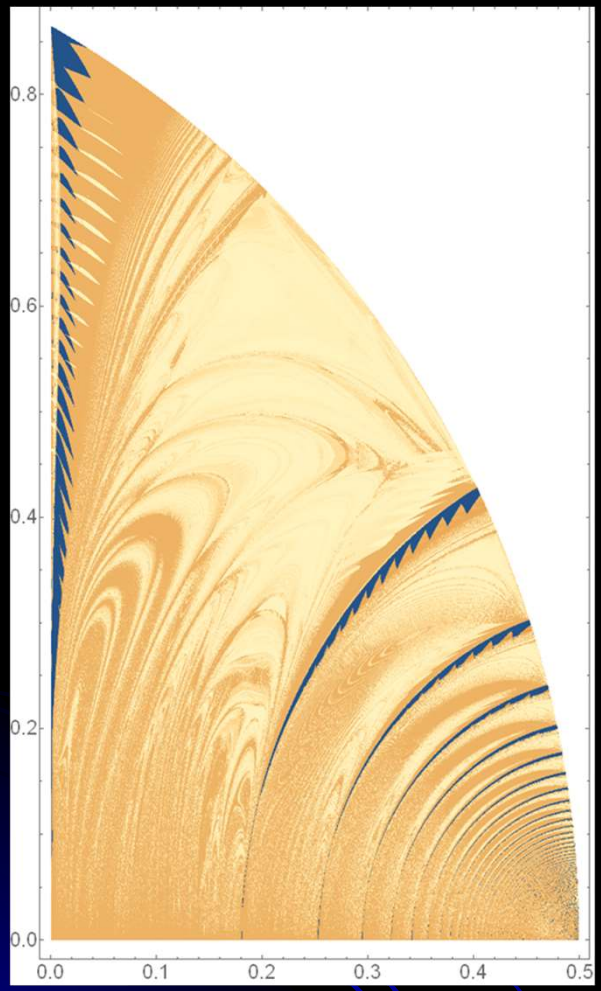
Screenshot 2019-04-12 04.31.12



Screenshot 2019-04-12 04.46.59

Kolmogorov complexity

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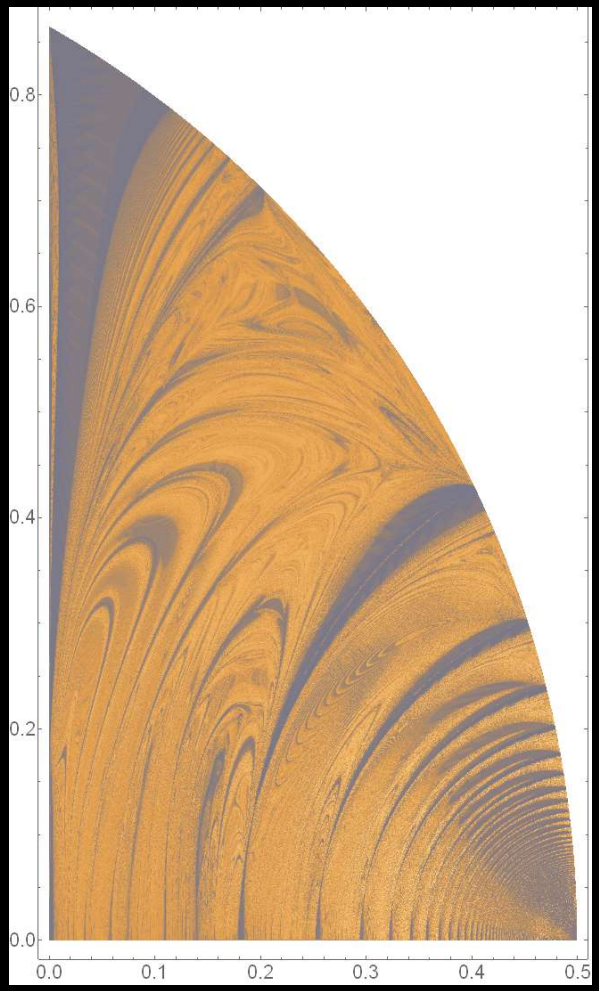
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0.1
0.2
0.3
0.4
0.5

Arnold complexity

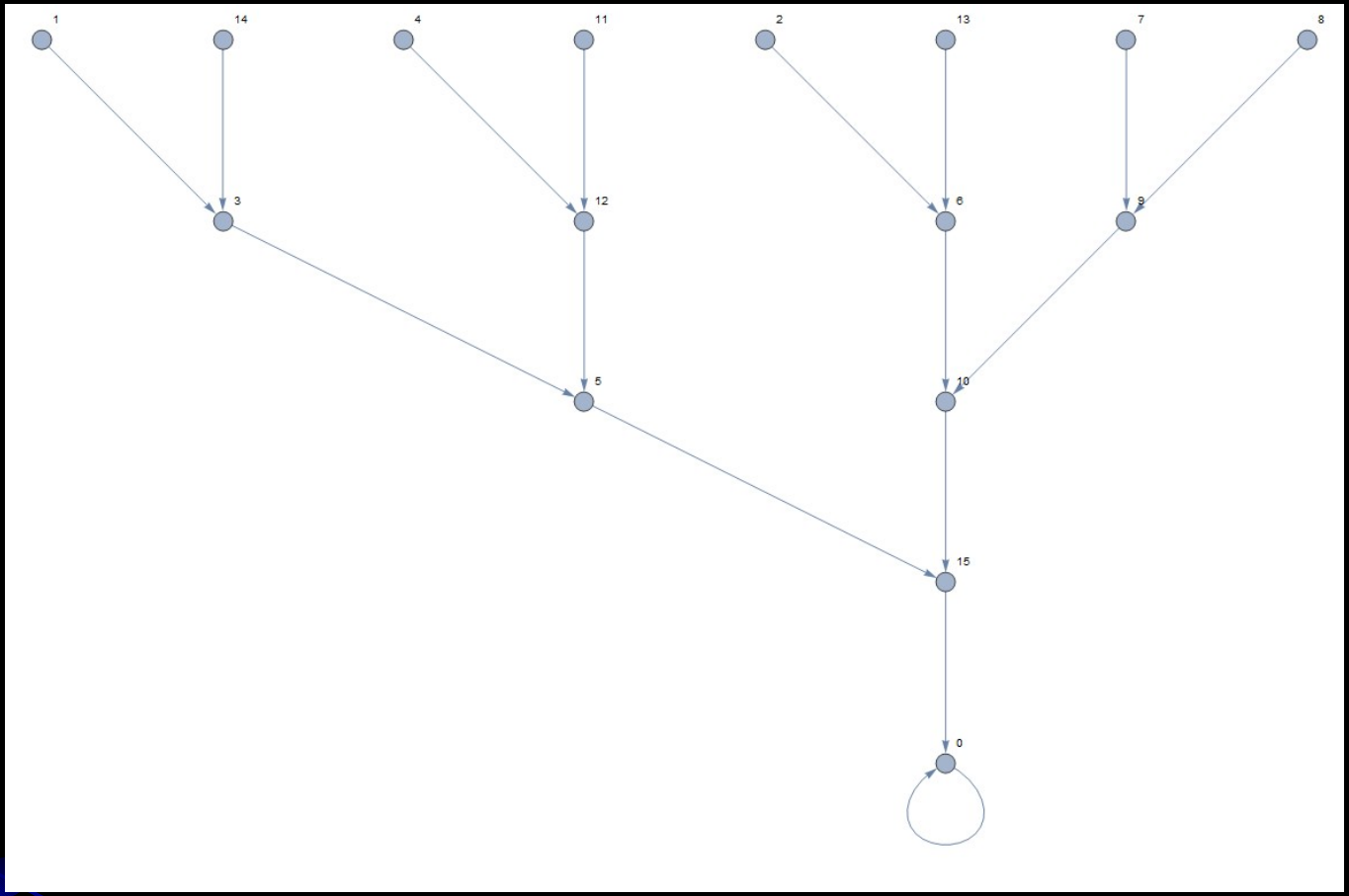
Consider a set M of all possible sequences of length n. Let us define (following Newton's idea) the increment sequence: we thus consider the linear operator $\boldsymbol{A}$: M $\rightarrow$ M,

$$y = \boldsymbol{A}x$$

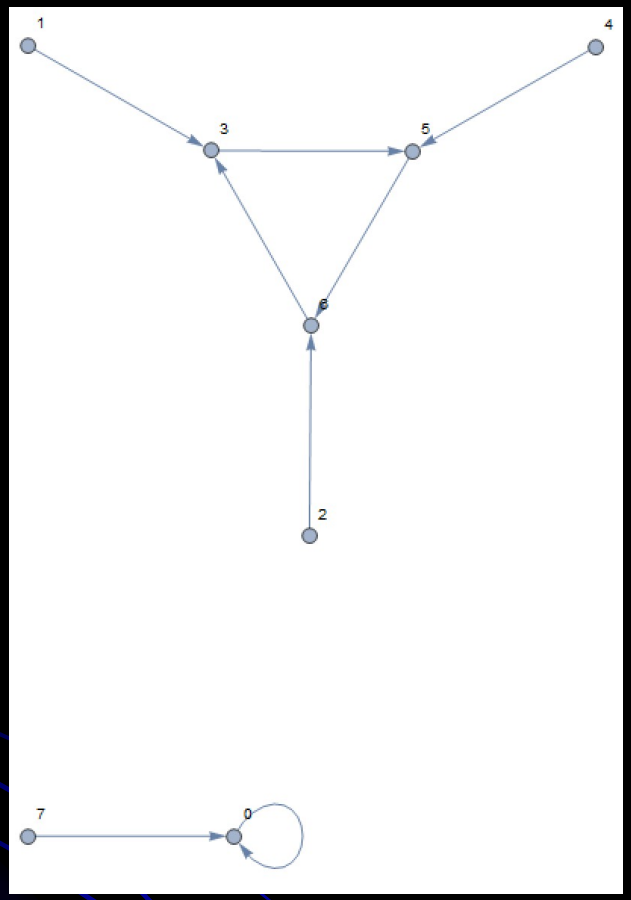
defined by the formula

$$y_j = x_{j+1} - x_j$$

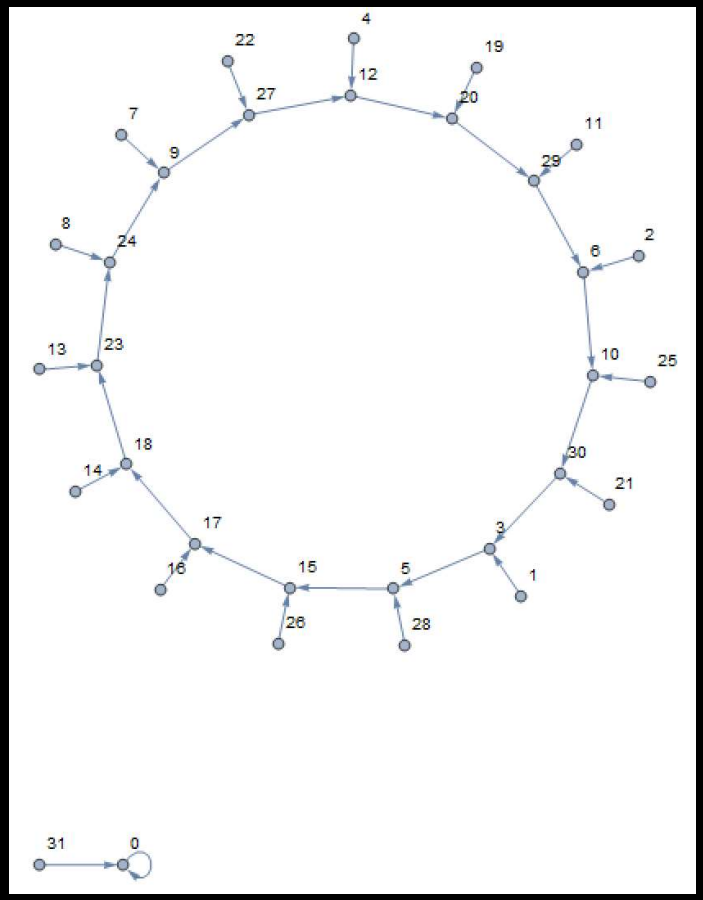
To have n increments, we define $x_{n+1} = x_1$, making our sequence x cyclic (the function x, whose value at j is x_j, is then n-periodic).



The map A of the finite set M into itself is described by a directed graph with 2^n vertices $x \in$ M. In this graph, exactly one edge starts from each vertex x (and leads to Ax).


1
4
3
5
6
2
7
0

$n = 3$


4
22
19
12
27
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7
11
9
29
8
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24
6
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23
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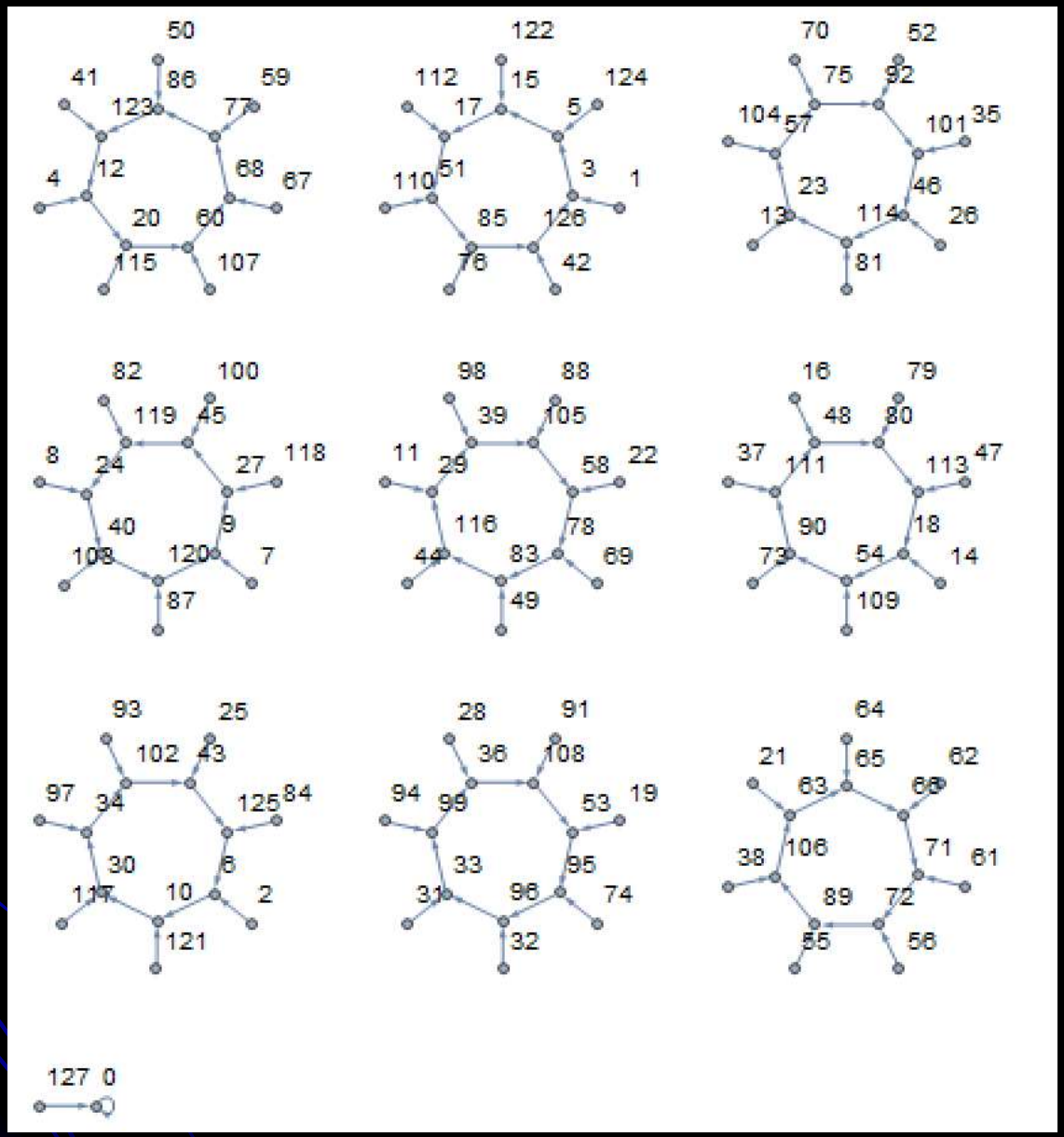
18
30
14
21
17
3
16
15
5
1
26
28
31
0

$n = 5$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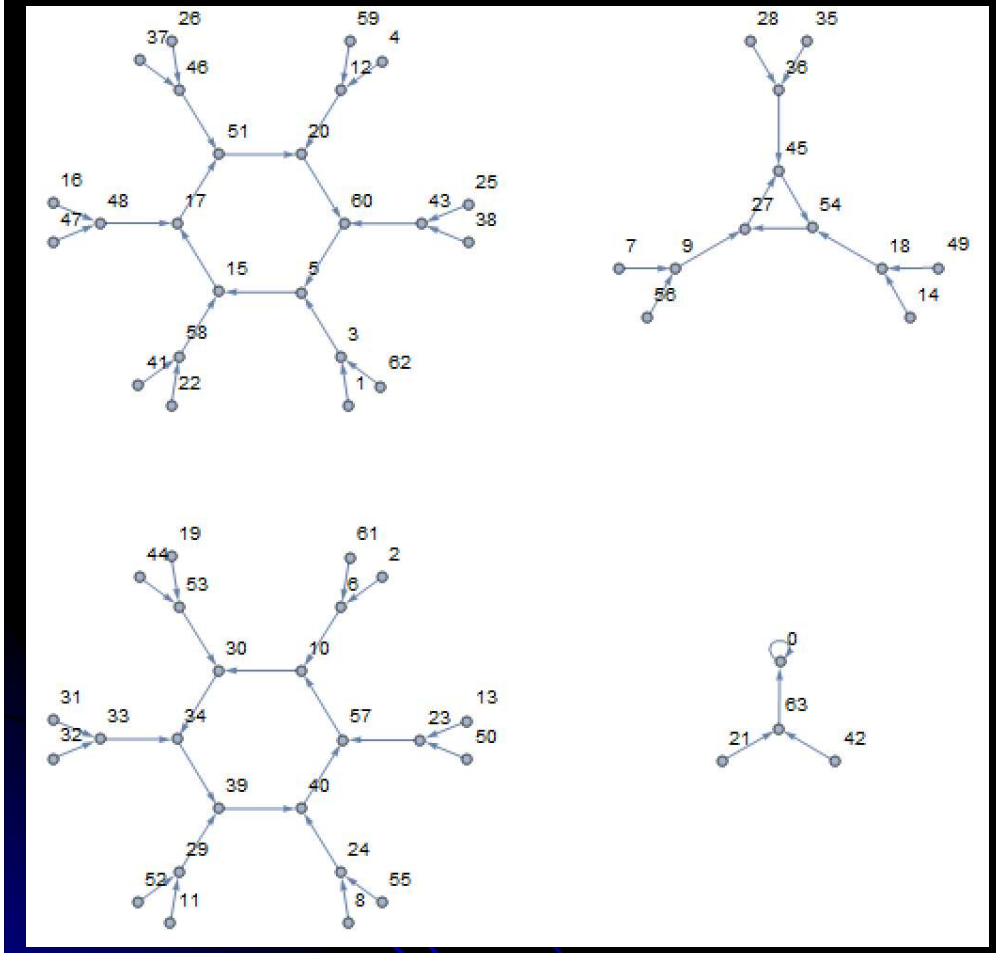
123 77
12 68 67
4
20 60
115 107
122
112 15 124
17 5
51 3 1
110
85 126
76 42
70 52
75 92
104 57 101 35
23 46
13 114 26
81
82 100
119 45
8 24 27 118
40 9
108 120 7
87
98 88
39 105
11 29 58 22
116 78
44 83 69
49
16 79
48 80
37 111 113 47
90 18
73 54 14
109
93 25
102 43
97 34 125 84
30 6
117 10 2
121
28 91
36 108
94 99 53 19
33 95
31 96 74
32
64
21 65 62
63 68
38 106 71 61
89 72
55 56
127 0

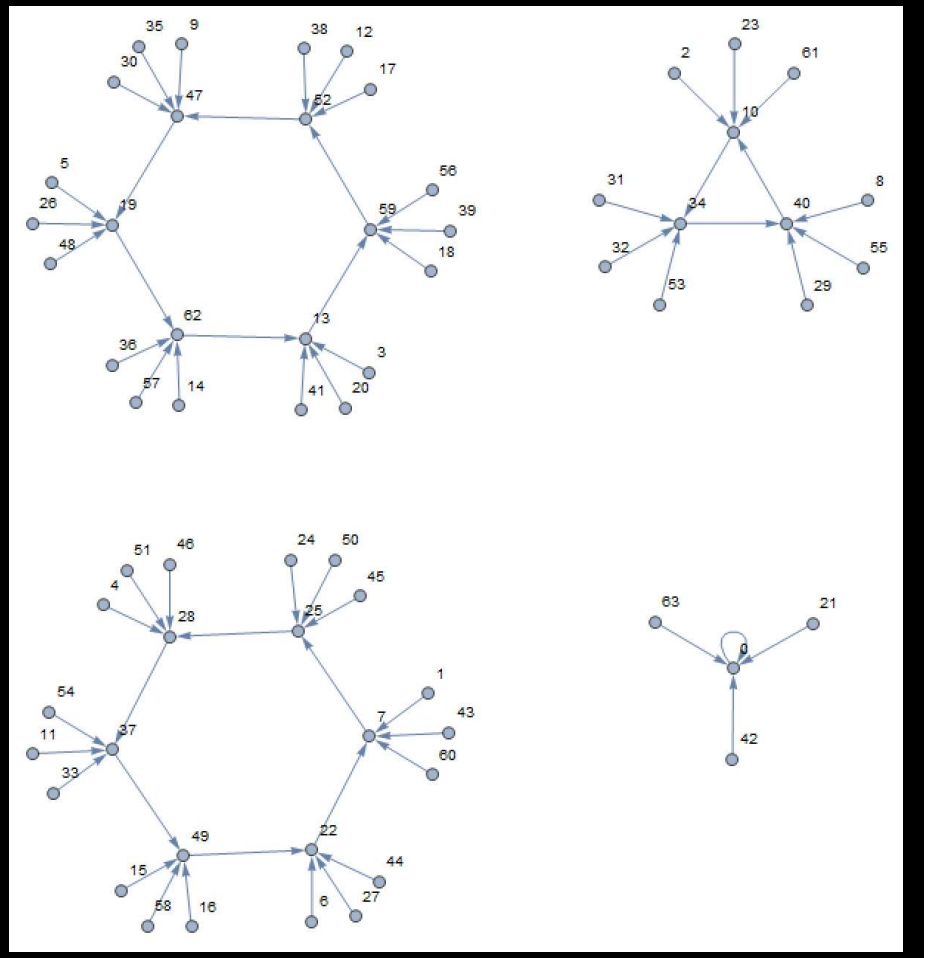
$n = 7$

binary



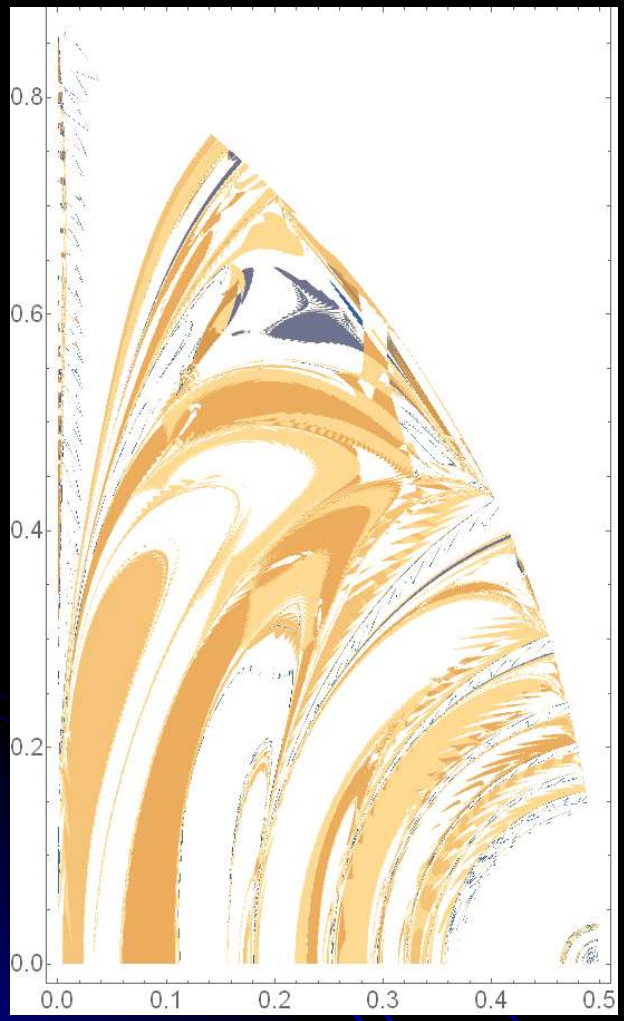
$n = 7$

ternary

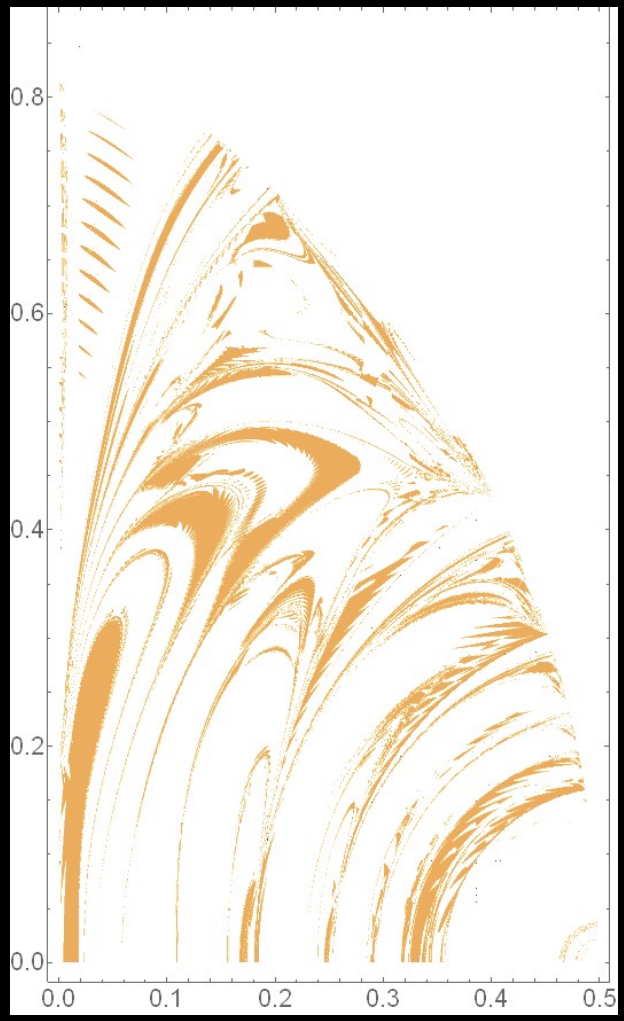


$n = 3$

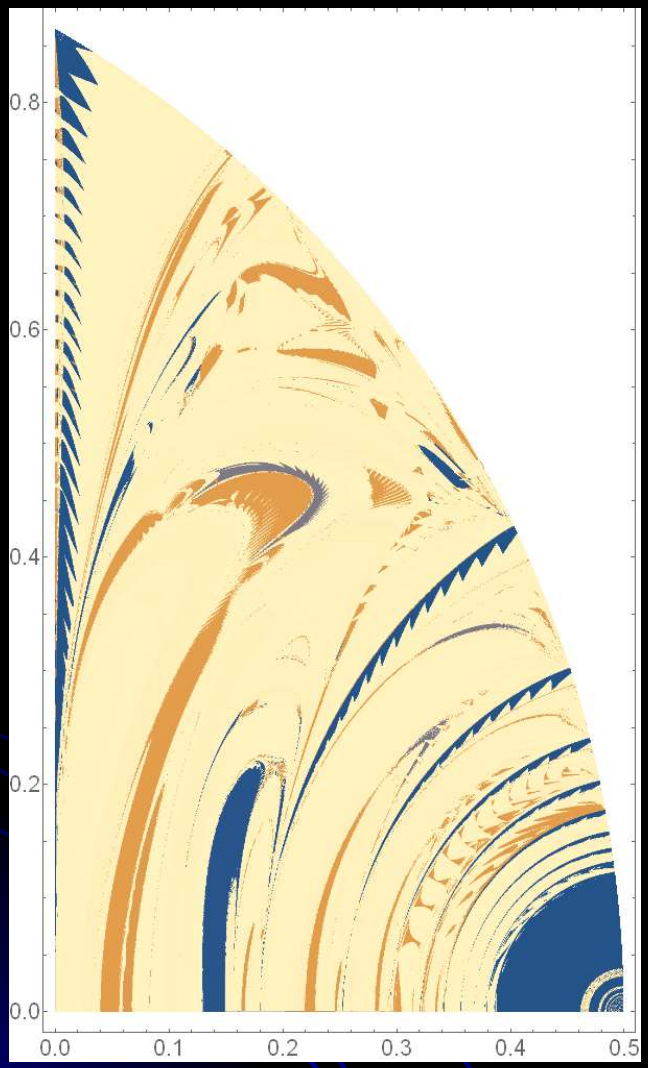
The de nition of the complexity: we say that an object x is more complicated if the length of the cycle of the component of the graph containing the point x is larger. Inside the components whose cycles have equal lengths, a vertex is said to be more complicated if its distance from the cycle is larger.



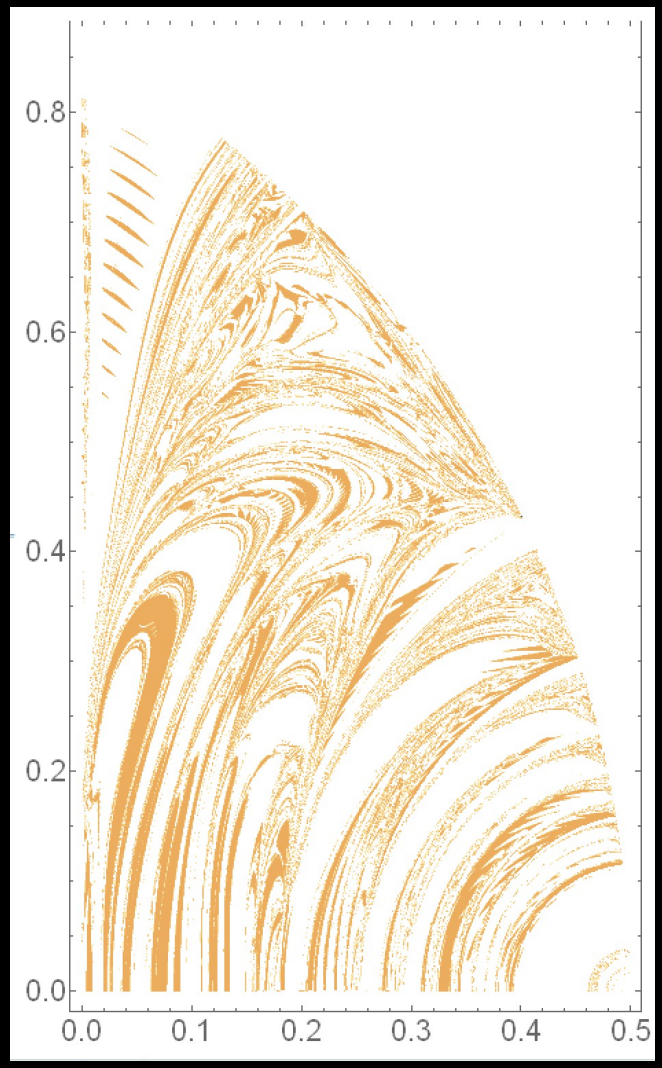
$n = 8$



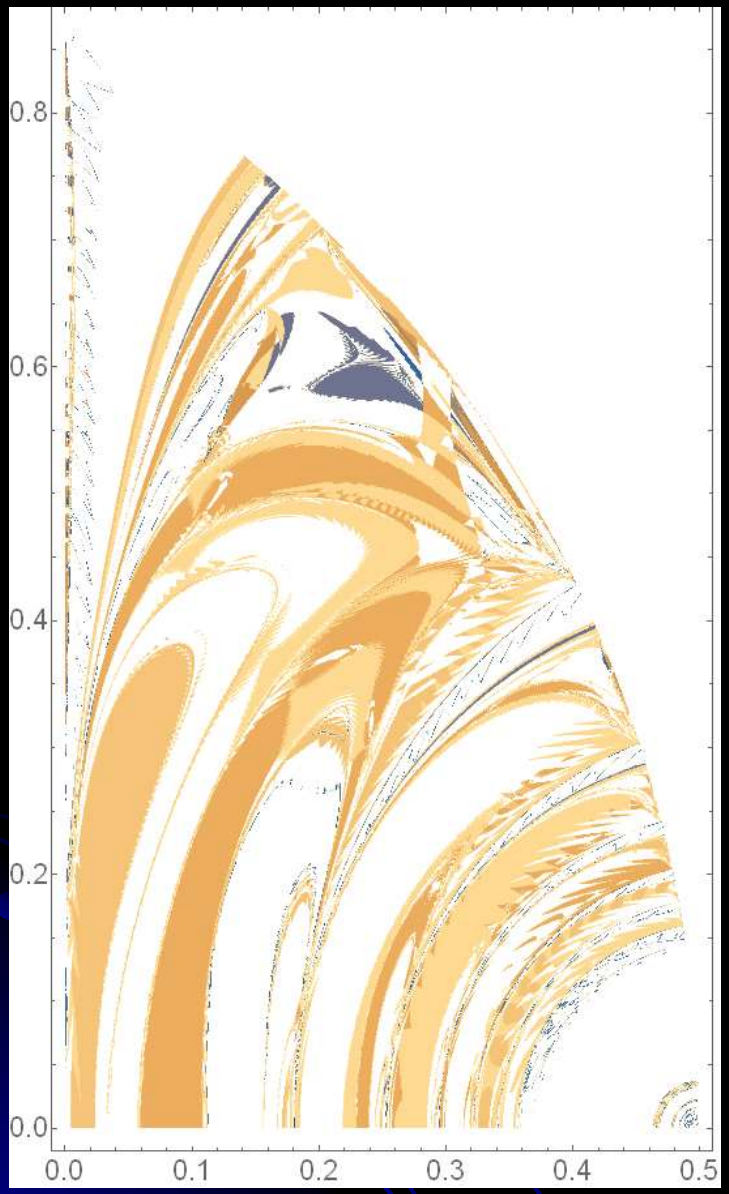
$n = 9$



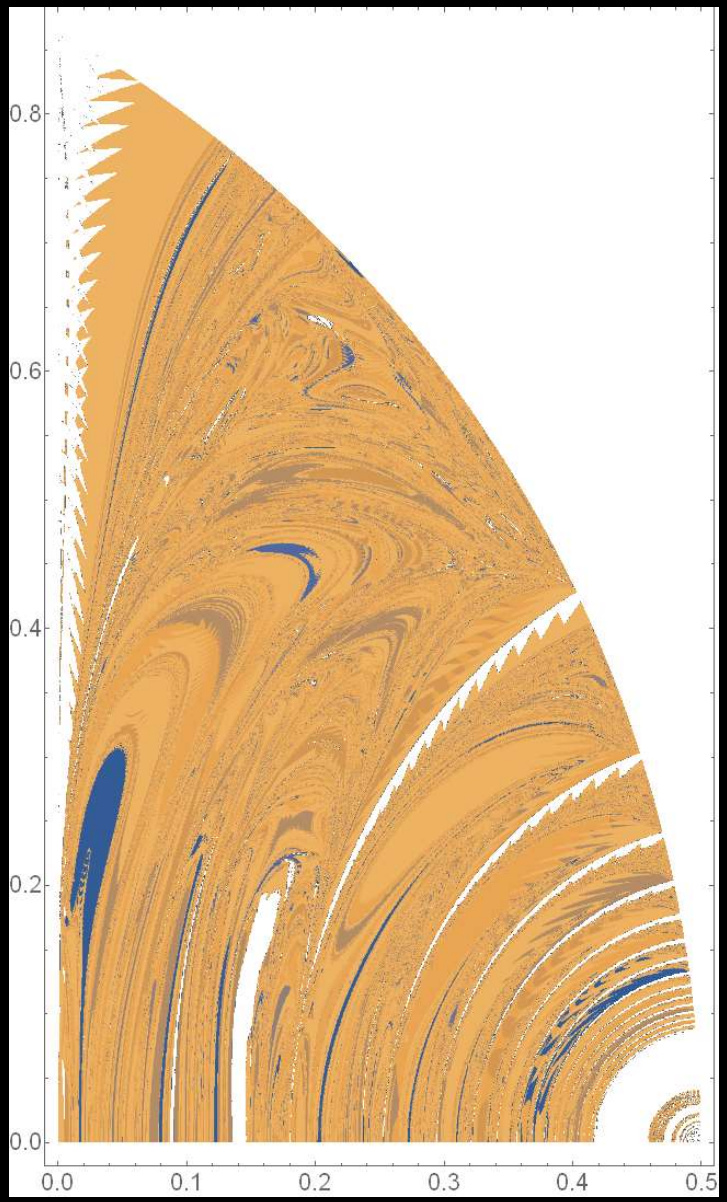
$n = 10$



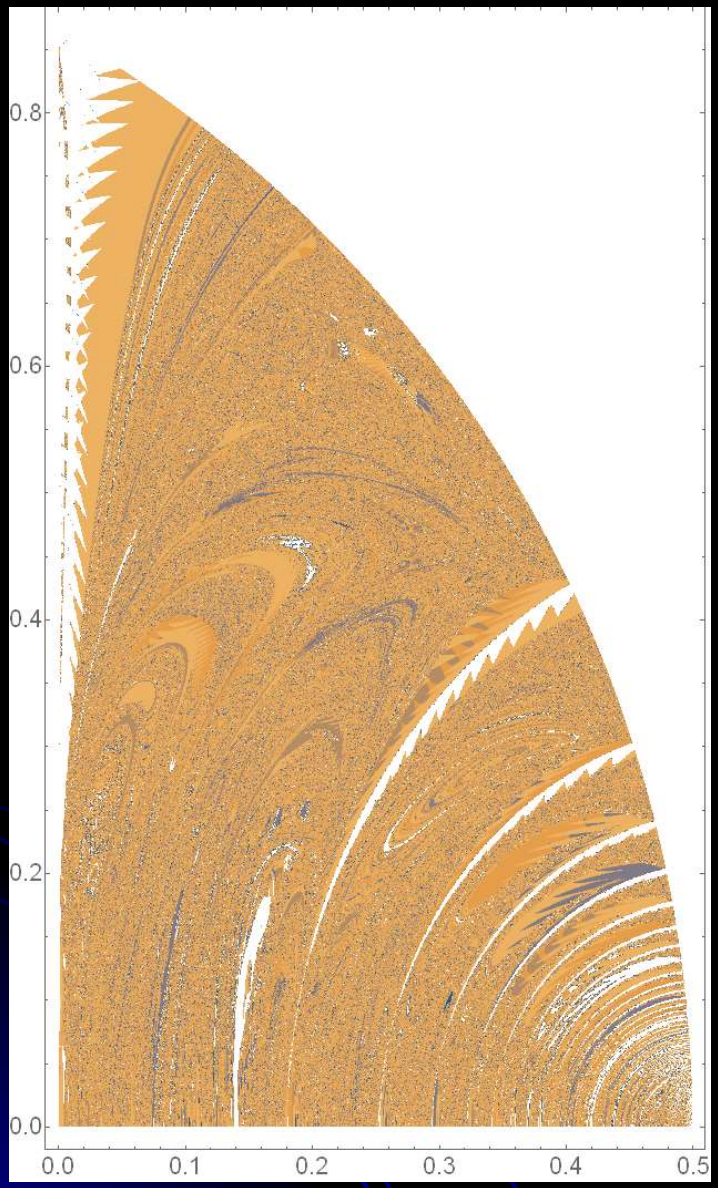
$n = 13$



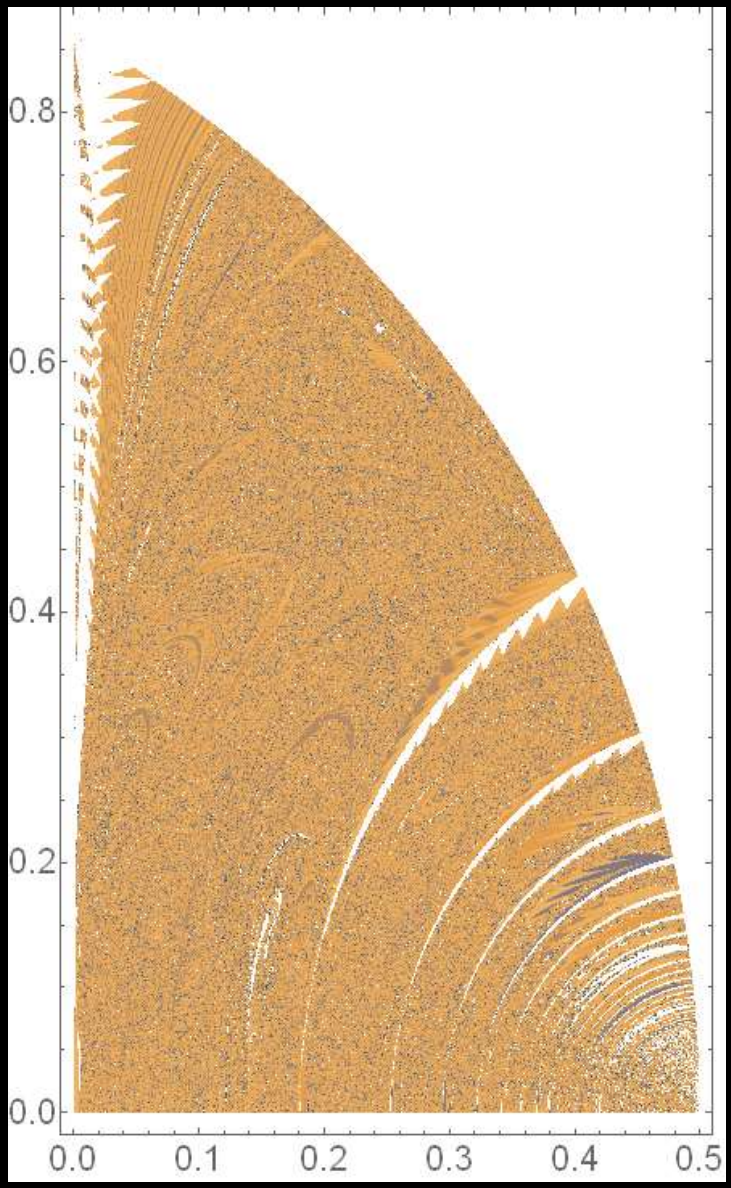
$n = 8$



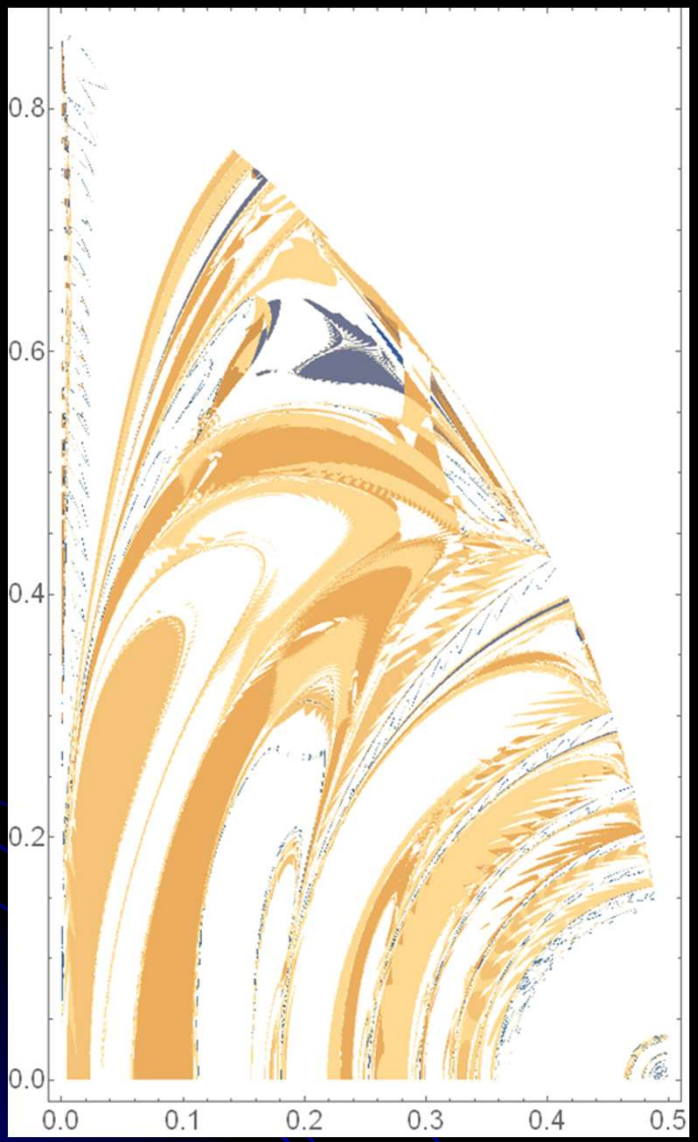
$n = 16$

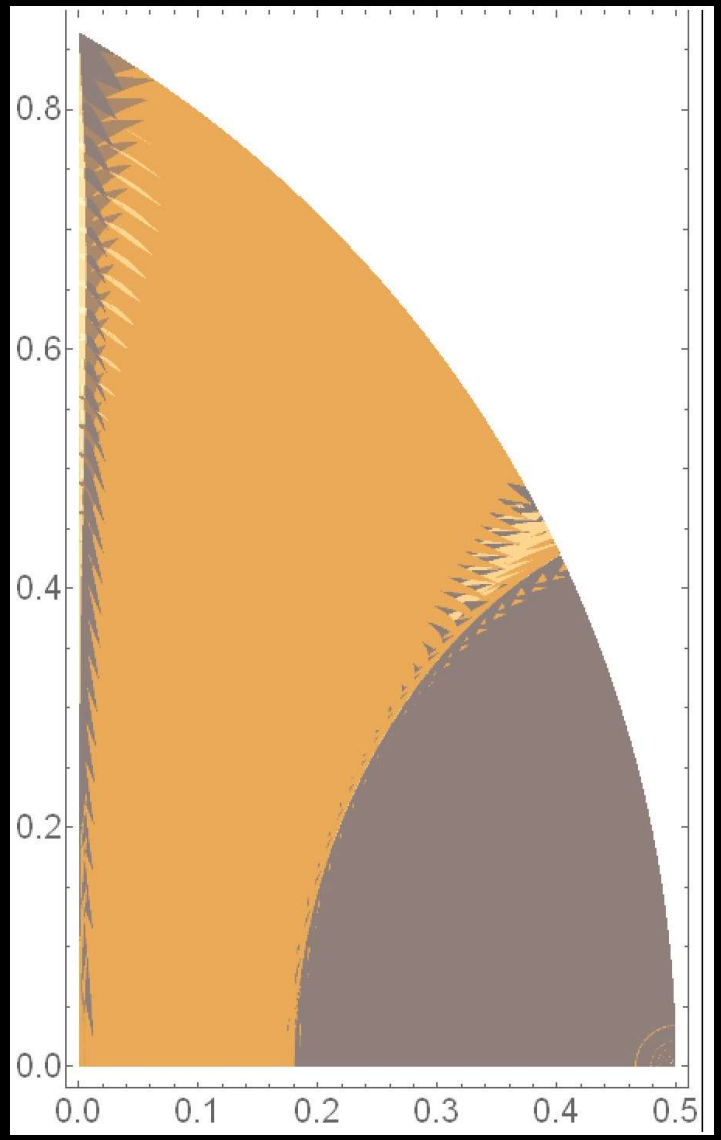


$n = 32$

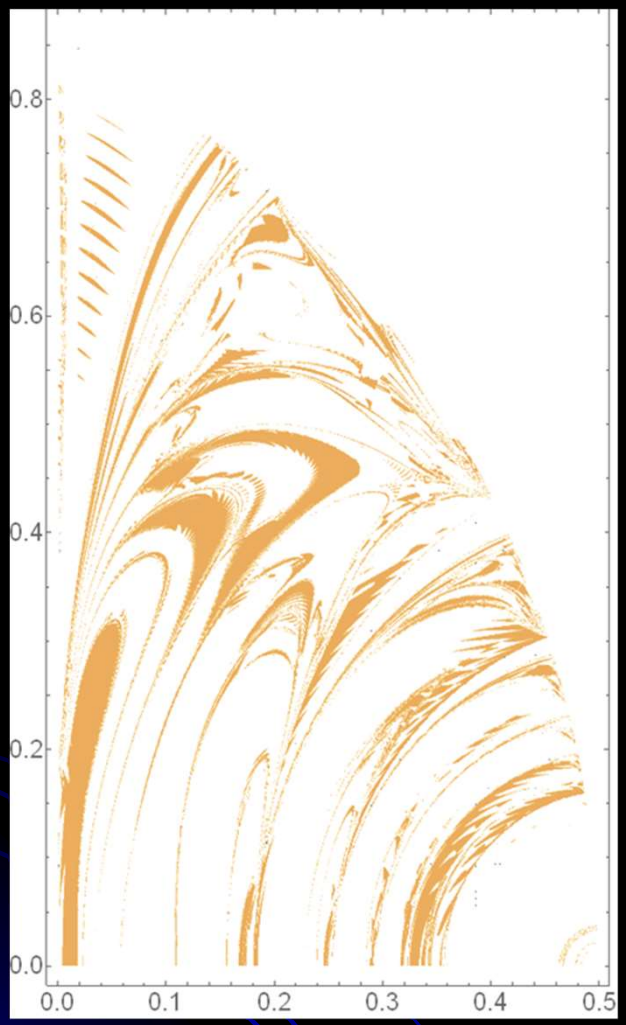


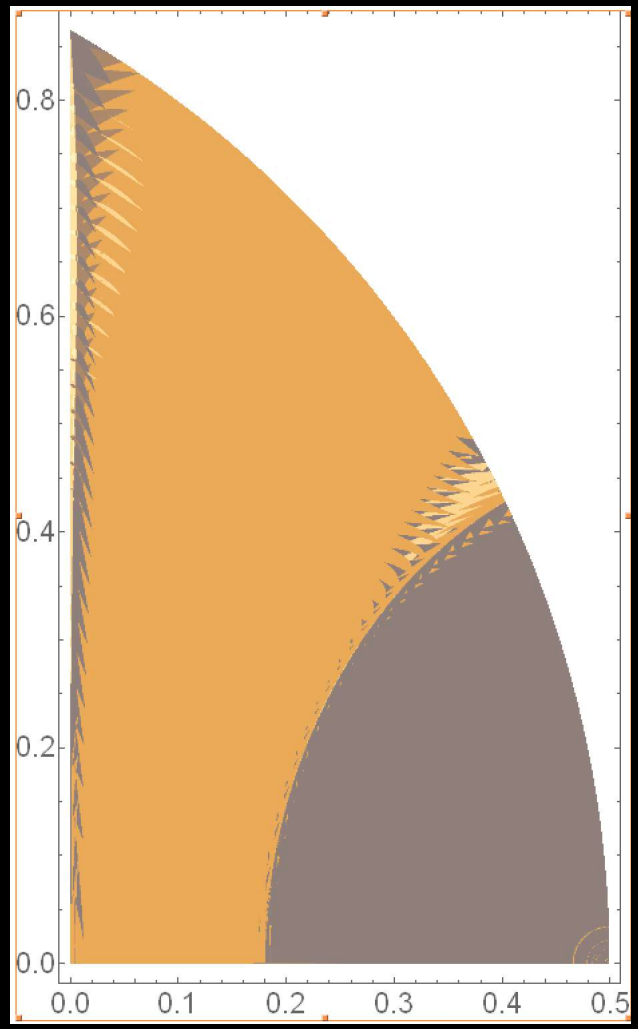
$n = 64$



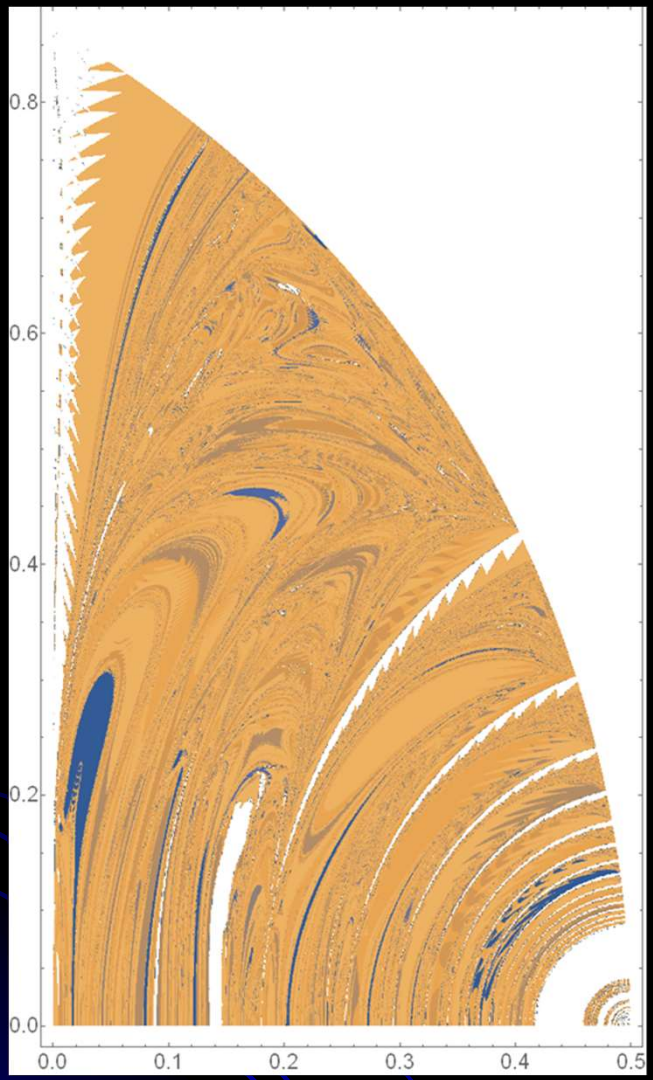


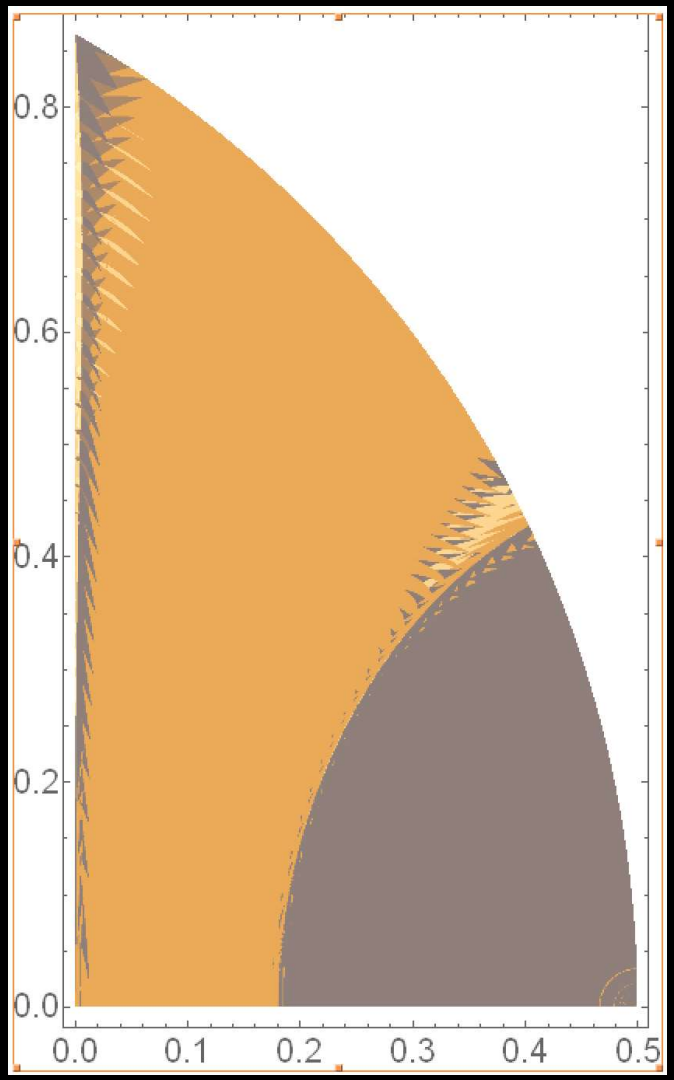
$n = 8$





$n = 9$





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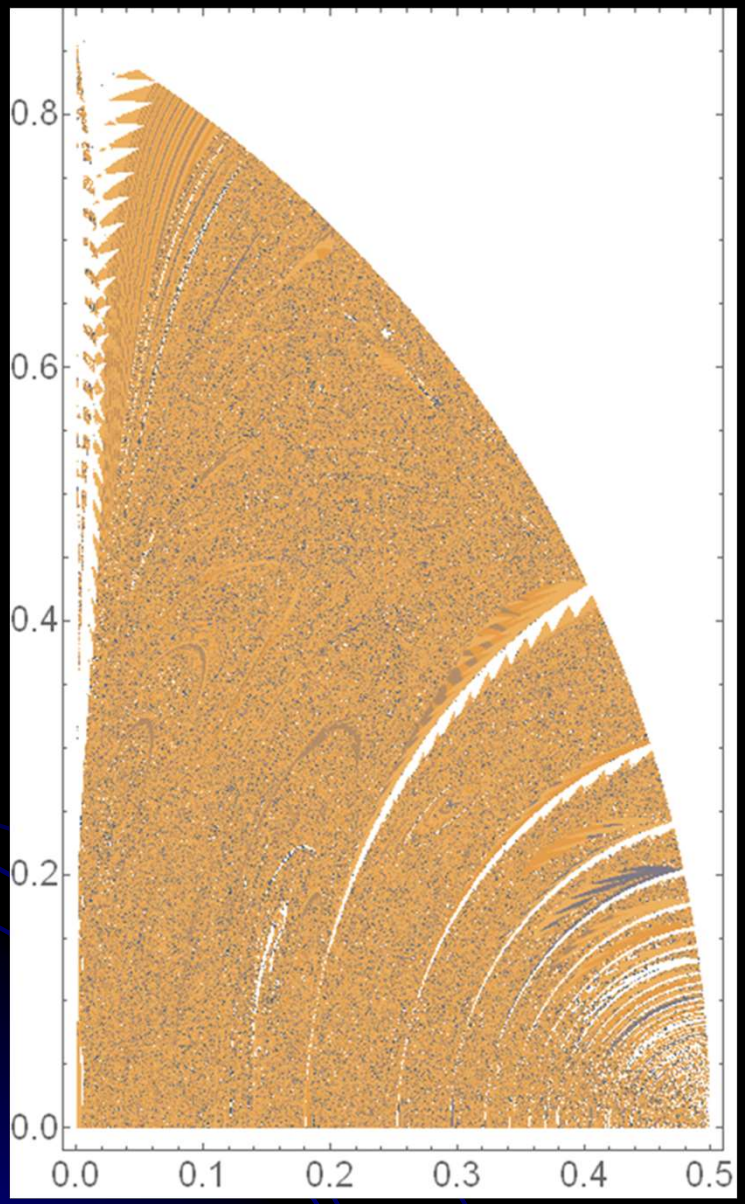
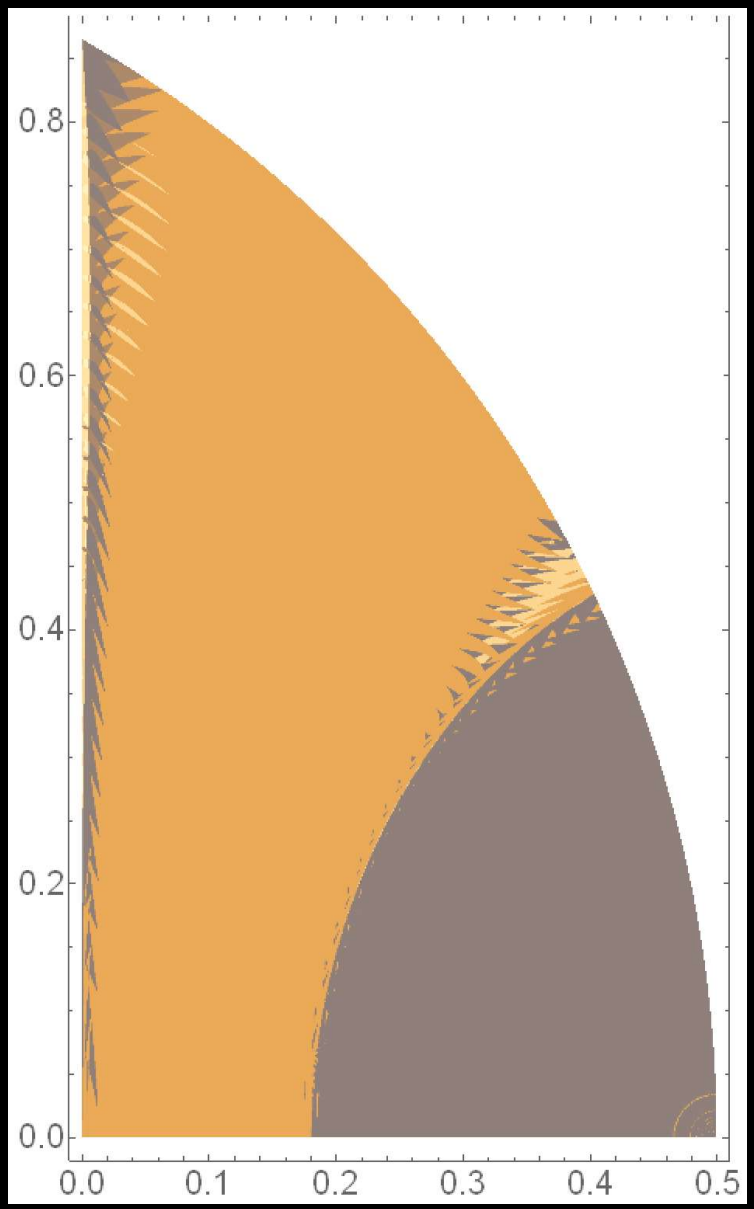
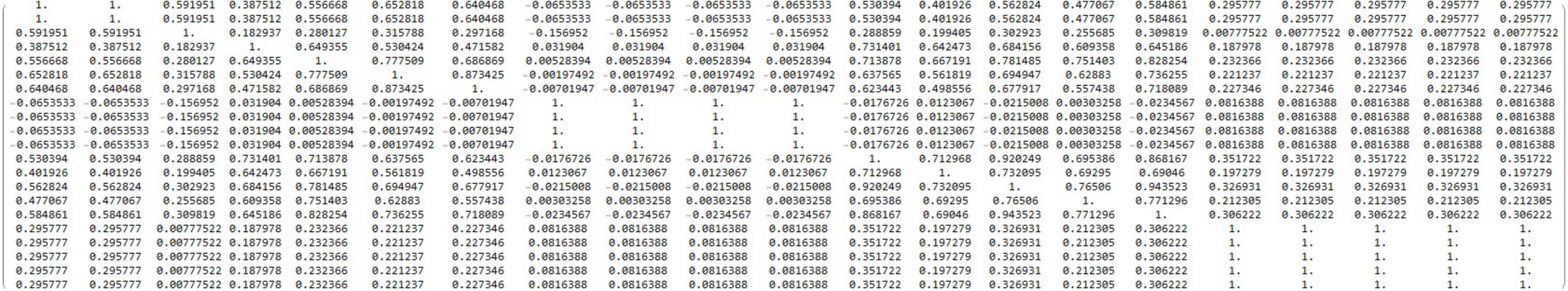




$n = 64$



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1.          1.          0.591951    0.387512    0.556668    0.652818    0.640468    -0.0653533  -0.0653533  -0.0653533  -0.0653533  0.530394    0.401926    0.562824    0.477067    0.584861    0.295777    0.295777    0.295777    0.295777    0.295777
0.591951    0.591951    1.          0.182937    0.280127    0.315788    0.297168    -0.156952   -0.156952   -0.156952   -0.156952   0.288859    0.199405    0.302923    0.255685    0.309819    0.00777522  0.00777522  0.00777522  0.00777522  0.00777522
0.387512    0.387512    0.182937    1.          0.649355    0.530424    0.471582    0.031904    0.031904    0.031904    0.031904    0.731401    0.642473    0.684156    0.609358    0.645186    0.187978    0.187978    0.187978    0.187978    0.187978
0.556668    0.556668    0.280127    0.649355    1.          0.777509    0.686869    0.00528394  0.00528394  0.00528394  0.00528394  0.713878    0.667191    0.781485    0.751403    0.828254    0.232366    0.232366    0.232366    0.232366    0.232366
0.652818    0.652818    0.315788    0.530424    0.777509    1.          0.873425    -0.00197492 -0.00197492 -0.00197492 -0.00197492 0.637565    0.561819    0.694947    0.62883     0.736255    0.221237    0.221237    0.221237    0.221237    0.221237
0.640468    0.640468    0.297168    0.471582    0.686869    0.873425    1.          -0.00701947 -0.00701947 -0.00701947 -0.00701947 0.623443    0.498556    0.677917    0.557438    0.718089    0.227346    0.227346    0.227346    0.227346    0.227346
-0.0653533  -0.0653533  -0.156952   0.031904    0.00528394  -0.00197492 -0.00701947 1.          1.          1.          1.          -0.0176726  0.0123067   -0.0215008  0.00303258  -0.0234567  0.0816388   0.0816388   0.0816388   0.0816388   0.0816388
-0.0653533  -0.0653533  -0.156952   0.031904    0.00528394  -0.00197492 -0.00701947 1.          1.          1.          1.          -0.0176726  0.0123067   -0.0215008  0.00303258  -0.0234567  0.0816388   0.0816388   0.0816388   0.0816388   0.0816388
-0.0653533  -0.0653533  -0.156952   0.031904    0.00528394  -0.00197492 -0.00701947 1.          1.          1.          1.          -0.0176726  0.0123067   -0.0215008  0.00303258  -0.0234567  0.0816388   0.0816388   0.0816388   0.0816388   0.0816388
-0.0653533  -0.0653533  -0.156952   0.031904    0.00528394  -0.00197492 -0.00701947 1.          1.          1.          1.          -0.0176726  0.0123067   -0.0215008  0.00303258  -0.0234567  0.0816388   0.0816388   0.0816388   0.0816388   0.0816388
0.530394    0.530394    0.288859    0.731401    0.713878    0.637565    0.623443    -0.0176726  -0.0176726  -0.0176726  -0.0176726  1.          0.712968    0.920249    0.695386    0.868167    0.351722    0.351722    0.351722    0.351722    0.351722
0.401926    0.401926    0.199405    0.642473    0.667191    0.561819    0.498556    0.0123067   0.0123067   0.0123067   0.0123067   0.712968    1.          0.732095    0.69295     0.69046     0.197279    0.197279    0.197279    0.197279    0.197279
0.562824    0.562824    0.302923    0.684156    0.781485    0.694947    0.677917    -0.0215008  -0.0215008  -0.0215008  -0.0215008  0.920249    0.732095    1.          0.76506     0.943523    0.326931    0.326931    0.326931    0.326931    0.326931
0.477067    0.477067    0.255685    0.609358    0.751403    0.62883     0.557438    0.00303258  0.00303258  0.00303258  0.00303258  0.695386    0.69295     0.76506     1.          0.771296    0.212305    0.212305    0.212305    0.212305    0.212305
0.584861    0.584861    0.309819    0.645186    0.828254    0.736255    0.718089    -0.0234567  -0.0234567  -0.0234567  -0.0234567  0.868167    0.69046     0.943523    0.771296    1.          0.306222    0.306222    0.306222    0.306222    0.306222
0.295777    0.295777    0.00777522  0.187978    0.232366    0.221237    0.227346    0.0816388   0.0816388   0.0816388   0.0816388   0.351722    0.197279    0.326931    0.212305    0.306222    1.          1.          1.          1.          1.
0.295777    0.295777    0.00777522  0.187978    0.232366    0.221237    0.227346    0.0816388   0.0816388   0.0816388   0.0816388   0.351722    0.197279    0.326931    0.212305    0.306222    1.          1.          1.          1.          1.
0.295777    0.295777    0.00777522  0.187978    0.232366    0.221237    0.227346    0.0816388   0.0816388   0.0816388   0.0816388   0.351722    0.197279    0.326931    0.212305    0.306222    1.          1.          1.          1.          1.
0.295777    0.295777    0.00777522  0.187978    0.232366    0.221237    0.227346    0.0816388   0.0816388   0.0816388   0.0816388   0.351722    0.197279    0.326931    0.212305    0.306222    1.          1.          1.          1.          1.
0.295777    0.295777    0.00777522  0.187978    0.232366    0.221237    0.227346    0.0816388   0.0816388   0.0816388   0.0816388   0.351722    0.197279    0.326931    0.212305    0.306222    1.          1.          1.          1.          1.
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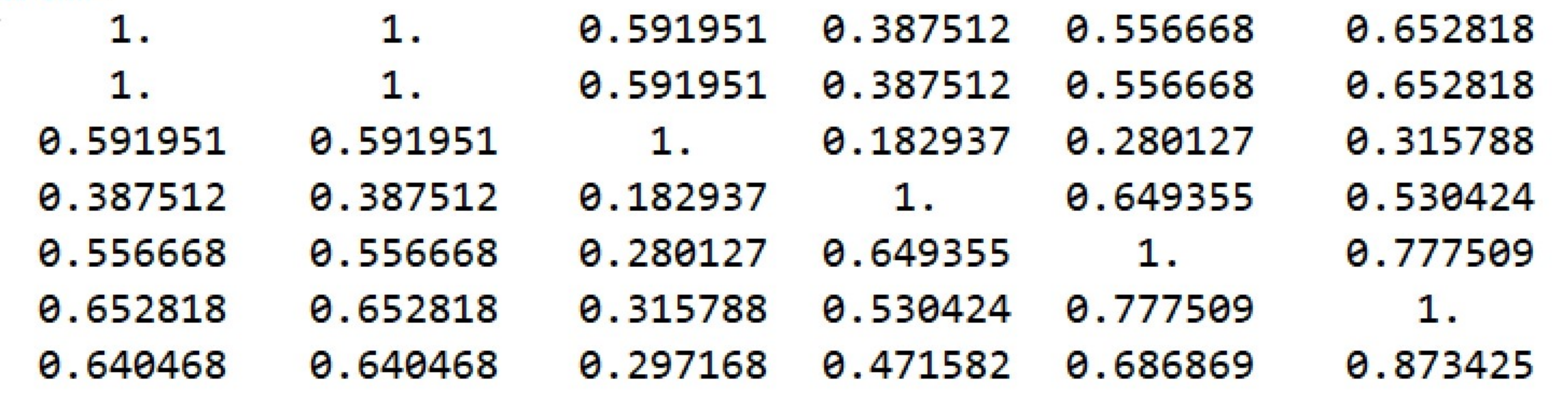

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   1.          1.        0.591951  0.387512  0.556668    0.652818
0.591951    0.591951        1.     0.182937  0.280127    0.315788
0.387512    0.387512     0.182937     1.     0.649355    0.530424
0.556668    0.556668     0.280127  0.649355     1.       0.777509
0.652818    0.652818     0.315788  0.530424  0.777509       1.
0.640468    0.640468     0.297168  0.471582  0.686869    0.873425
```

Thank you for attention!