Invariant Projectors in Wreath Product Representations

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Abstract. An algorithm for computing the complete set of irreducible invariant projectors in the space of the permutation representation of a wreath product is described. This set provides the irreducible decomposition of the representation. The corresponding C program constructs decompositions of representations of high dimensions and high ranks.

1. Introduction

A description of a physical system commonly involves a *space* X, on which a group of *spatial symmetries* G (or $G(X)$) acts, and a set of *local states* V with a group of *local symmetries* F (or $F(V)$). X, V and F can be treated, respectively, as the *base*, the *typical fiber* and the *structure group* of a *fiber bundle*. A state of the whole system is a function from X to V, i.e., a *section* of the bundle. A natural symmetry group that acts on the set of sections V^X and preserves the structure of the bundle is the *wreath product* [1, 2] of F and G

$$\widetilde{W} = F \wr G \cong F^X \rtimes G\,. \tag{1}$$

The action of $\widetilde{W}$ on V^X is defined by

$$v(x)\left(f(x)\,,g\right) = v\left(xg^{-1}\right) f\left(xg^{-1}\right),$$

where $v \in V^X$, $f \in F^X$, $g \in G$; the *right-action* convention is used for all group actions. Importance of wreath products:

1. The *universal embedding theorem* (Kaloujnine-Krasner) states that any extension of group A by group B is isomorphic to a subgroup of $A \wr B$, i.e., the wreath product is a universal object containing all extensions.
2. Classification of *maximal subgroups* of the *symmetric* group (the *O'Nan-Scott theorem*) essentially involves wreath products [3].
3. The wreath product $\mathsf{S}_m \wr \mathsf{S}_n$ is the automorphism group of the *hypercubic graph* or *Hamming scheme* $H(n,m)$ in coding theory [4].
4. Unitary representations of wreath products arise naturally in the study of multipartite quantum systems.

The main step in the study of group representations is to decompose them into irreducible components. Our algorithm [5] decomposes representations of finite groups via computing a *complete set of mutually orthogonal irreducible invariant projectors*. A similar construction in ring theory is called a *complete set of primitive* *orthogonal idempotents*. An arbitrary ring with such a set can be represented as a direct sum of indecomposable rings. This is called a *Peirce decomposition* [6, 7]. In our case, *irreducible invariant projectors* are *primitive idempotents* of the *centralizer ring* of a group representation. The dimension of this ring is called the *rank of the representation*. The program in [5] proved to be very effective in problems with low ranks. In particular, it coped with many high dimensional representations of simple groups and their "small" extensions (which typically have low ranks), presented in the ATLAS [8]. However, wreath products, which contain all possible extensions, are far from simple groups and their representations have high ranks. The approach proposed here allows us to decompose wreath product representations with very high dimensions and ranks.

2. Centralizer Ring of Wreath Product Representation

We assume that $X \cong \{1, \ldots, N\}$ and $V \cong \{1, \ldots, M\}$, and hence $G(X) \leq \mathsf{S}_N$ and $F(V) \leq \mathsf{S}_M$. The permutation representation $\widetilde{P}$ of $\widetilde{W}$ is defined by $(0,1)$-matrices of the size $M^N \times M^N$ that have the form

$$\widetilde{P}(\widetilde{w})_{u,v} = \delta_{u\widetilde{w},v}, \text{ where } \widetilde{w} \in \widetilde{W};\ u, v \in V^X;\ \delta \text{ is the } \textit{Kronecker delta.}$$

As a representation space, we assume an M^N-dimensional Hilbert space $\widetilde{\mathcal{H}}$ over some abelian extension of $\mathbb{Q}$ being a splitting field for the local group F. We denote the rank of the representation $\widetilde{P}$ by $\widetilde{R}$, and we denote the basis of the centralizer ring by $\widetilde{A}_1, \ldots, \widetilde{A}_{\widetilde{R}}$. The basis elements are solutions of the system of equations

$$\widetilde{P}\left(\widetilde{w}^{-1}\right) \widetilde{A} \widetilde{P}(\widetilde{w}) = \widetilde{A},\ \widetilde{w} \in \widetilde{W}\,. \tag{2}$$

A more detailed analysis of (2), taking into account the structure of the wreath product (1), allows to obtain explicit expressions for the basis elements of the centralizer ring of $\widetilde{P}$

$$\widetilde{A}_r = \sum_{q \in rG} A_{q_1} \otimes \cdots \otimes A_{q_N}\,. \tag{3}$$

Here

1. R and $A_1, \ldots, A_R$ are, respectively, the rank and the basis of the centralizer ring for the M-dimensional permutation representation of the local group F.
2. $r \in \overline{R}^X$ denotes a mapping from X into $\overline{R} = \{1, \ldots, R\}$.
3. rG is the G-orbit of the mapping $r \in \overline{R}^X$ with respect to the action defined by $rg = [r_{1g}, \ldots, r_{Ng}]$ for $g \in G$. The notation $r = [r_1, \ldots, r_N]$ is assumed.

It is easy to verify that the basis elements (3) form a complete system, i.e.,

$$\sum_{i=1}^{\widetilde{R}} \widetilde{A}_{r^{(i)}} = \mathbb{J}_{M^N},$$

where $\mathbb{J}_{M^N}$ is the $M^N \times M^N$ *all-ones matrix*, $r^{(i)}$ denotes some numbering of the orbits of G on $\overline{R}^X$.

3. Complete Set of Irreducible Orthogonal Invariant Projectors

The complete set of irreducible orthogonal invariant projectors is a subset of the centralizer ring, specified by the conditions of idempotency and mutual orthogonality. Using the properties of the tensor (Kronecker) product [9], their consequences and some additional technical considerations we come to the following.

Let $B_1, \ldots, B_K$ be the complete set of irreducible orthogonal projectors in the permutation representation of the local group F. Let $\overline{K} = \{1, \ldots, K\}$ and $\overline{K}^X$ be the set of all mappings from X into $\overline{K}$. The action of $g \in G$ on the mapping $k \in \overline{K}^X$ is defined as $kg = [k_{1g}, \ldots, k_{Ng}]$. Then we have

Proposition. *The irreducible orthogonal invariant projector in the permutation representation of the wreath product takes the form*

$$\widetilde{B}_k = \sum_{\ell \in kG} B_{\ell_1} \otimes \cdots \otimes B_{\ell_N}\,, \qquad (4)$$

where kG denotes the G-orbit of the mapping k on the set $\overline{K}^X$.

The easily verifiable completeness condition $\sum_{i=1}^{\widetilde{K}} \widetilde{B}_{k^{(i)}} = \mathbb{1}_{M^N}$ holds. Here $\widetilde{K}$ is the number of irreducible components of the wreath product representation, $\mathbb{1}_{M^N}$ is the identity matrix in the representation space, $k^{(i)}$ denotes some numbering of the orbits of G on $\overline{K}^X$.

To compute the basis elements (3) of the centralizer ring and projectors (4), we wrote a program in C. The input data for the program are the generators of the spatial and local groups, and the complete set of irreducible invariant projectors of the local group (obtained, for example, by the program described in [5]).

4. Calculation Example

We give here the calculation for the representation of the wreath product of the rotational symmetry groups of the octahedron and icosahedron. The dimension and rank are $M^N = 2\,176\,782\,336$ and $\widetilde{R} = 122\,776$.

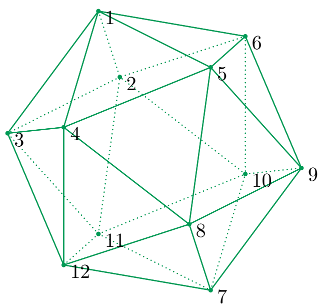

FIGURE 1. Icosahedron.

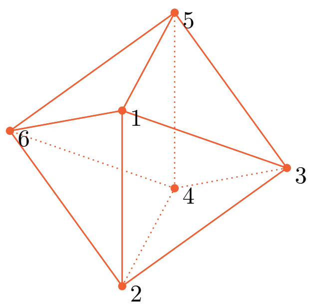

FIGURE 2. Octahedron.

Space Group. The space X is the icosahedron whose vertices form the set of points: $X \cong \{1, \dots, 12\}$, see Fig. 1. As a group of spatial symmetries we take the group A_5, which describes the *rotational* (or *chiral*) *symmetries* of the icosahedron. For the vertex numbering as in Fig. 1, the space symmetry group can be generated by two permutations:

$$G(X) = \big\langle (1,7)(2,8)(3,12)(4,11)(5,10)(6,9),\ (2,3,4,5,6)(8,9,10,11,12) \big\rangle \cong \mathsf{A}_5.$$

Local Group. The local states, $V \cong \{1, \dots, 6\}$, are the vertices of the octahedron. The group of rotational symmetries of the octahedron is S_4. For the vertex numbering of Fig. 2, the local symmetry group has the following presentation by two generators

$$F(V) = \big\langle (1,3,5)\,(2,4,6)\,,\ (1,2,4,5) \big\rangle \cong \mathsf{S}_4.$$

The six-dimensional permutation representation $\underline{\mathbf{6}}$ of $F(V)$ has rank 3, and the basis of the centralizer ring is

$$A_1 = \mathbb{1}_6,\ A_2 = \begin{pmatrix} \mathbb{0}_3 & \mathbb{1}_3 \\ \mathbb{1}_3 & \mathbb{0}_3 \end{pmatrix},\ A_3 = \begin{pmatrix} Y & Y \\ Y & Y \end{pmatrix},\ \text{where } Y = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \qquad (5)$$

The irreducible decomposition of the representation is $\underline{\mathbf{6}} = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3}$. The complete set of primitive orthogonal idempotents can be written in the basis (5) as follows

$$B_{\mathbf{1}} = \frac{1}{6}\left(A_1 + A_2 + A_3\right), B_{\mathbf{2}} = \frac{1}{3}\left(A_1 + A_2 - \frac{1}{2}A_3\right), B_{\mathbf{3}} = \frac{1}{2}\left(A_1 - A_2\right). \qquad (6)$$

Program Output. The calculation was performed on PC with 3.30GHz CPU and 16GB RAM. The superscripts in the list of '`Irreducible dimensions`' represent the numbers of equal dimensions. The expressions for primitive idempotents are tensor product polynomials in the matrices (6). '`Checksum`' is the sum of all dimensions, which should coincide with the dimension of the representation.

Wreath product $\mathsf{S}_4(\textit{octahedron}) \wr \mathsf{A}_5(\textit{icosahedron})$
`Representation dimension:` **2 176 782 336**
`Rank:` **122 776**
`Wreath product decomposition is multiplicity free`
`Number of irreducible components:` 122 776
`Number of different dimensions:` 134
`Irreducible dimensions:`

$1, 4^{6}, 6^{3}, 8^{6}, 9^{3}, 12^{15}, 16^{32}, 18^{7}, 20, 24^{70}, 32^{41}, 36^{86}, 45, 48^{191}, 54^{26}, 64^{84}, 72^{298}, 80^{4}, 81^{7}, 96^{412}, 108^{223}, 128^{114}, 144^{913}, 162^{54}, 180^{8}, 192^{704}, 216^{926}, 243^{4}, 256^{104}, 288^{1804}, 320^{7}, 324^{504}, 384^{772}, 405^{4}, 432^{2517}, 486^{99}, 512^{76}, 576^{2508}, 648^{1909}, 720^{17}, 729^{9}, 768^{705}, 864^{4303}, 972^{818}, 1024^{51}, 1152^{2562}, 1280^{3}, 1296^{4455}, 1458^{141}, 1536^{479}, 1620^{16}, 1728^{5322}, 1944^{2712}, 2048^{20}, 2187^{4}, 2304^{1935}, 2592^{6708}, 2880^{14}, 2916^{961}, 3072^{223}, 3456^{4575}, 3645^{7}, 3888^{5495}, 4096^{4}, 4374^{136}, 4608^{1004}, 5120, 5184^{6924}, 5832^{2754}, 6144^{59}, 6480^{18}, 6561^{9}, 6912^{2719}, 7776^{\underline{\mathbf{6966}}}, 8192^{3}, 8748^{822}, 9216^{329}, 10368^{4760}, 11520^{10}, 11664^{4695}, 12288^{19}, 13122^{98}, 13824^{1011}, 14580^{13}, 15552^{5781}, 17496^{1999}, 18432^{83}, 19683^{3}, 20736^{2085}, 23328^{4826}, 25920^{16}, 26244^{511}, 27648^{260}, 31104^{2964},$

$32805^{3}, 34992^{2775}, 36864^{5}, 39366^{55}, 41472^{534}, 46080, 46656^{3012}, 52488^{1023}, 55296^{15}, 58320^{19}, 59049^{5}, 62208^{877}, 69984^{2173}, 78732^{242}, 82944^{48}, 93312^{1038}, 103680^{4}, 104976^{1079}, 118098^{27}, 124416^{102}, 131220^{8}, 139968^{905}, 157464^{355}, 186624^{130}, 209952^{568}, 233280^{7}, 236196^{84}, 279936^{148}, 295245, 314928^{254}, 354294^{6}, 419904^{116}, 472392^{79}, 524880^{3}, 531441, 629856^{62}, 708588^{15}, 944784^{26}, 1180980,$
$\underline{\mathbf{1417176}}^{9}$

`Checksum =` 2176782336 `Maximum number of equal dimensions =` 6966

`Wreath irreducible projectors:`

$$
\begin{aligned}
\widetilde{B}_1 =& B_1^{\otimes 12}\\
\widetilde{B}_2 =& B_1^{\otimes 3} \otimes B_2 \otimes B_1^{\otimes 6} \otimes B_2 \otimes B_1\\
\widetilde{B}_3 =& B_1^{\otimes 9} \otimes B_2 \otimes B_1^{\otimes 2} + B_1^{\otimes 4} \otimes B_2 \otimes B_1^{\otimes 7}\\
&\vdots\\
\widetilde{B}_{61387} =& B_2 \otimes B_3 \otimes B_1 \otimes B_2^{\otimes 2} \otimes B_1^{\otimes 3} \otimes B_3^{\otimes 2} \otimes B_2 \otimes B_3\\
&+ B_3 \otimes B_2 \otimes B_1 \otimes B_2 \otimes B_1 \otimes B_3 \otimes B_1 \otimes B_2 \otimes B_3 \otimes B_1 \otimes B_3 \otimes B_2\\
&+ B_1^{\otimes 2} \otimes B_3 \otimes B_2 \otimes B_3^{\otimes 2} \otimes B_2 \otimes B_3 \otimes B_1 \otimes B_2^{\otimes 2} \otimes B_1\\
&+ B_1 \otimes B_2^{\otimes 2} \otimes B_3 \otimes B_1 \otimes B_3^{\otimes 2} \otimes B_2 \otimes B_3 \otimes B_1 \otimes B_2 \otimes B_1\\
\widetilde{B}_{61388} =& B_2^{\otimes 2} \otimes B_3 \otimes B_1^{\otimes 2} \otimes B_3 \otimes B_1 \otimes B_2 \otimes B_3 \otimes B_2^{\otimes 2} \otimes B_3\\
&+ B_1 \otimes B_2 \otimes B_3 \otimes B_2^{\otimes 2} \otimes B_3 \otimes B_2^{\otimes 2} \otimes B_3 \otimes B_1^{\otimes 2} \otimes B_3\\
\widetilde{B}_{61389} =& B_1^{\otimes 2} \otimes B_2^{\otimes 3} \otimes B_1 \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_1 \otimes B_3^{\otimes 2}\\
&+ B_2 \otimes B_1 \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_1 \otimes B_3 \otimes B_2 \otimes B_1^{\otimes 2} \otimes B_2 \otimes B_3\\
&+ B_1 \otimes B_3 \otimes B_2 \otimes B_1^{\otimes 3} \otimes B_3 \otimes B_2^{\otimes 3} \otimes B_3 \otimes B_2\\
&+ B_3 \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_1^{\otimes 2} \otimes B_2 \otimes B_1^{\otimes 2} \otimes B_2 \otimes B_3 \otimes B_2\\
&+ B_2 \otimes B_3^{\otimes 3} \otimes B_1 \otimes B_2 \otimes B_1^{\otimes 3} \otimes B_2^{\otimes 3}\\
&+ B_3 \otimes B_2^{\otimes 2} \otimes B_3 \otimes B_2^{\otimes 2} \otimes B_1 \otimes B_3 \otimes B_1^{\otimes 3} \otimes B_2\\
&\vdots\\
\widetilde{B}_{122774} =& B_3^{\otimes 2} \otimes B_2 \otimes B_3^{\otimes 9} + B_3^{\otimes 3} \otimes B_2 \otimes B_3^{\otimes 8} + B_3^{\otimes 10} \otimes B_2 \otimes B_3 + B_3^{\otimes 11} \otimes B_2\\
\widetilde{B}_{122775} =& B_3^{\otimes 2} \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_3^{\otimes 7} + B_3^{\otimes 2} \otimes B_2 \otimes B_3^{\otimes 2} \otimes B_2 \otimes B_3^{\otimes 6}\\
&+ B_3^{\otimes 3} \otimes B_2 \otimes B_3^{\otimes 4} \otimes B_2 \otimes B_3^{\otimes 3} + B_3^{\otimes 5} \otimes B_2 \otimes B_3^{\otimes 4} \otimes B_2 \otimes B_3\\
&+ B_3^{\otimes 8} \otimes B_2 \otimes B_3^{\otimes 2} \otimes B_2 + B_3^{\otimes 9} \otimes B_2 \otimes B_3 \otimes B_2\\
\widetilde{B}_{122776} =& B_3^{\otimes 3} \otimes B_2^{\otimes 2} \otimes B_3^{\otimes 7} + B_2 \otimes B_3^{\otimes 4} \otimes B_2 \otimes B_3^{\otimes 6}\\
&+ B_3 \otimes B_2 \otimes B_3^{\otimes 3} \otimes B_2 \otimes B_3^{\otimes 6} + B_3^{\otimes 6} \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_3^{\otimes 3}\\
&+ B_3^{\otimes 7} \otimes B_2^{\otimes 2} \otimes B_3^{\otimes 3} + B_3^{\otimes 9} \otimes B_2^{\otimes 2} \otimes B_3
\end{aligned}
$$

```
Time: 0.58 sec
Maximum number of tensor monomials: 531441
```

For comparison, we give a part of the output for a somewhat larger problem.

Wreath product $\mathsf{A}_5(icosahedron) \wr \mathsf{A}_5(icosahedron)$
`Representation dimension:` **8 916 100 448 256**
`Rank:` **3 875 157**
`Wreath product decomposition is multiplicity free`
`Number of irreducible components:` 3 875 157
`Number of different dimensions:` 261

`Time: 7.35 sec`
`Maximum number of tensor monomials: 16777216`

5. Conclusion

One of the main goals of the work was to develop a tool for the study of models of multipartite quantum systems. The projection operators obtained by the program are matrices of huge dimension. Obviously, the explicit calculation of such matrices is impossible. However, the expression of projectors for wreath products in the form of tensor polynomials makes it possible to reduce the computation of quantum correlations to a sequence of computations with small matrices of local projectors.

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