Developing a Wolfram Demonstrations Project for bifurcation diagrams of nonlinear ODE

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Field of study

- $y''_{xx} + \lambda f\big(y(x)\big) = 0,\ where\ x \in (0,1)$
- $y'_x(0) = 0,\ \ y(1) = 0$

- $\lambda > 0$ – bifurcation parameter
- $f(y)$ – nonlinear function
  - $f(y) = (y - a_1)(y - a_2)(y - a_3) \dots (y - a_{2n-1})(a_{2n} - y)$

- bifurcation curve and diagram
- P. Korman - Y. Li - T. Ouyang Theorem

?

- Exact number of positive solutions
- How the solutions change due changing the bifurcation parameter λ

BVP

- $y''_{xx} + \lambda f\big(x, y(x)\big) = 0$, $where\ x \in (a, b)$, λ>0
- $y(a) = y(b) = 0$

Solutions:

- $y = y(x, \lambda)$

?

- Exact number of positive solutions
- How the solutions change due changing the bifurcation parameter λ

Autonomous 2nd order ODE

- $y''_{xx} + \lambda f\big(y(x)\big) = 0, \, where \; x \in (a, b)$
- $y(a) = y(b) = 0$

Positive solutions

- $y''_{xx} + \lambda f\big(y(x)\big) = 0, \, where \; x \in (-1,1)$
- $y(-1) = y(1) = 0$

Even number of zeros of solutions

- $y''_{xx} + \lambda f\big(y(x)\big) = 0, \, where \; x \in (0,1)$
- $y'_x(0) = 0, \;\; y(1) = 0$

Korman P., Global solution branches and exact multiplicity of solutions for two point boundary value problems.

Bifurcation curve $\lambda(a)$

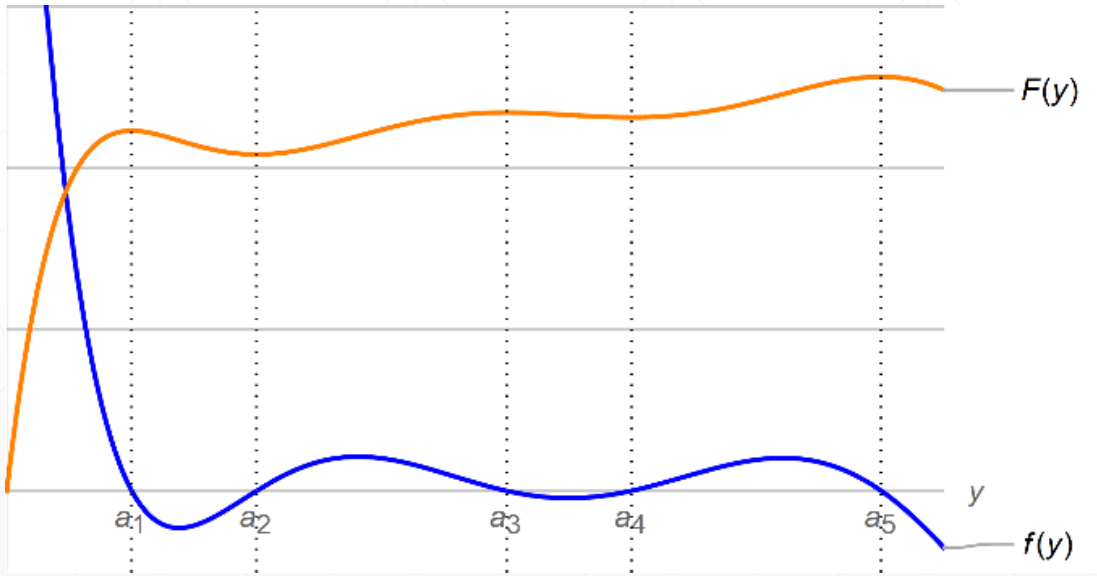
- $\lambda = \lambda(a)$ - bifurcation **curve**;
- Turning points of $\lambda(a)$ – bifurcation **points**;
  - $\lambda'(a) = 0$
- Plot of function $\lambda = \lambda(a)$ – bifurcation **diagram**, implying an image of the change in the possible dynamic modes of the system with a change in the value of bifurcation parameter λ.

$$\lambda(a) = \frac{1}{2}\left(\int_0^a \frac{dt}{\sqrt{F(a)-F(y)}}\right)^2$$ - bifurcation curve

$f(y)$ as a polynomial of odd degree

- $f(y) = (y - a_1)(y - a_2)(y - a_3) \dots (y - a_{2n-2})(a_{2n-1} - y)$
  - $0 \le a_1 < a_2 < \dots < a_{2n-2} < a_{2n-1}$
  - $F(a_1) < F(a_2) \dots < F(a_{2n-2}) < F(a_{2n-1})$

$$y''_{xx} + \lambda f(y) = 0, \; where \;\; x \in (0,1)$$
$$y'_x(0) = 0, \;\; y(1) = 0$$



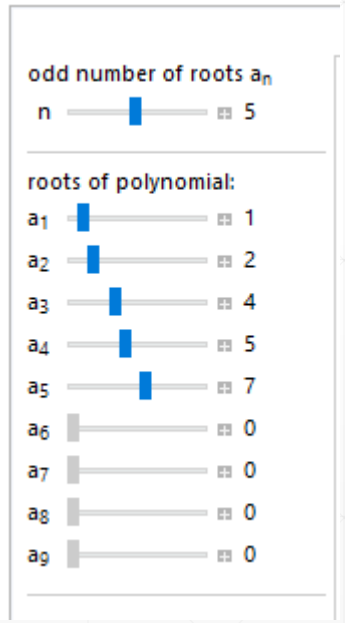
- Trivial solutions of BVP:
  $$\boldsymbol{y = a_i}$$
- $f(y) > 0$ *on* $(a_{2n-2}; a_{2n-1})$
- $f(y) < 0$ *on* $(a_{2n-3}; a_{2n-2})$

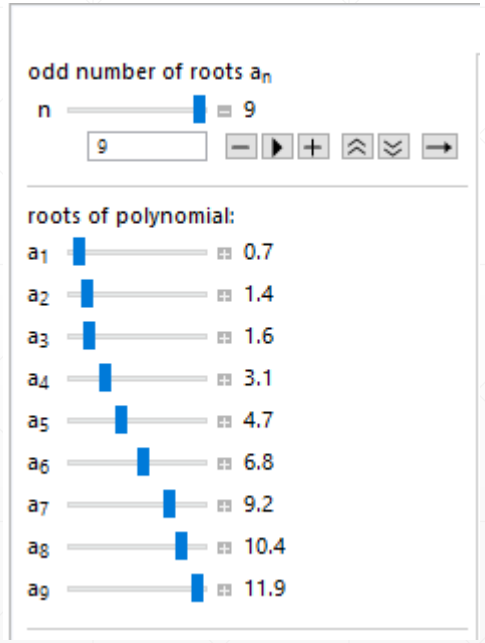
Wolfram Demonstrations Project (WDP)

- open-source collection of compact interactive programs
- hosted by Wolfram Research
- work with free Wolfram CDF Player
- Wolfram Demonstrations Project community
- WDP
  - title
  - manipulate code (incl. initialization)
  - caption
  - thumbnail

Developing a Manipulate code

- Manipulate command = set of control elements (sliders):
  - Odd number of roots n ($n = 5$ by default);
  - Roots of polynomial $f(y) - a_i$.

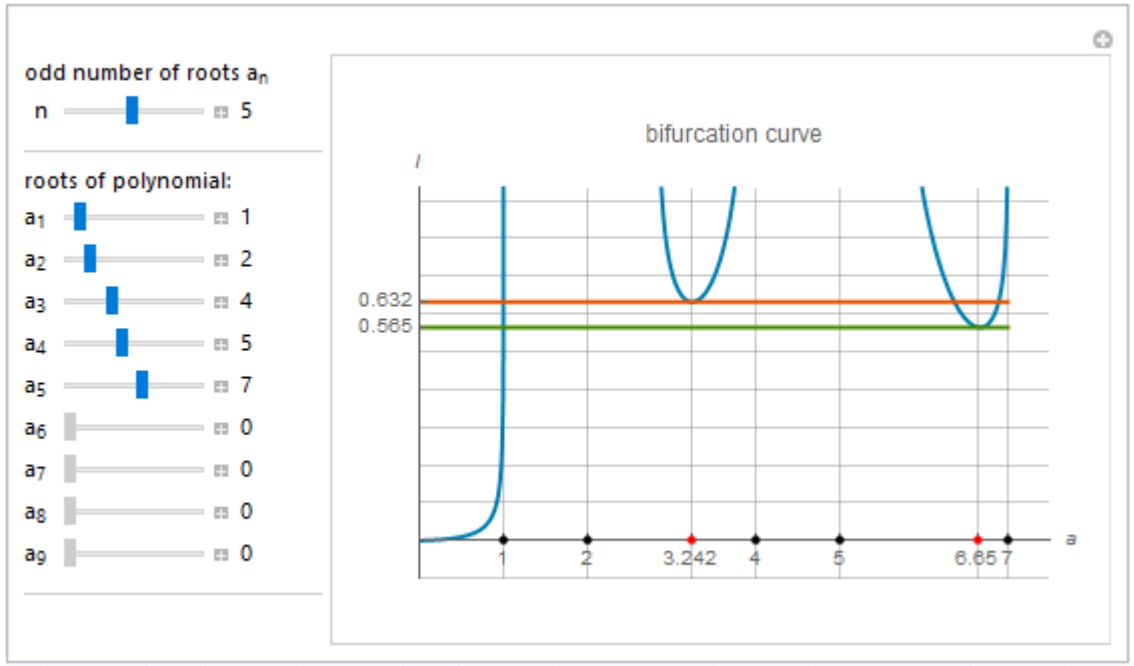




Developing a Manipulate code

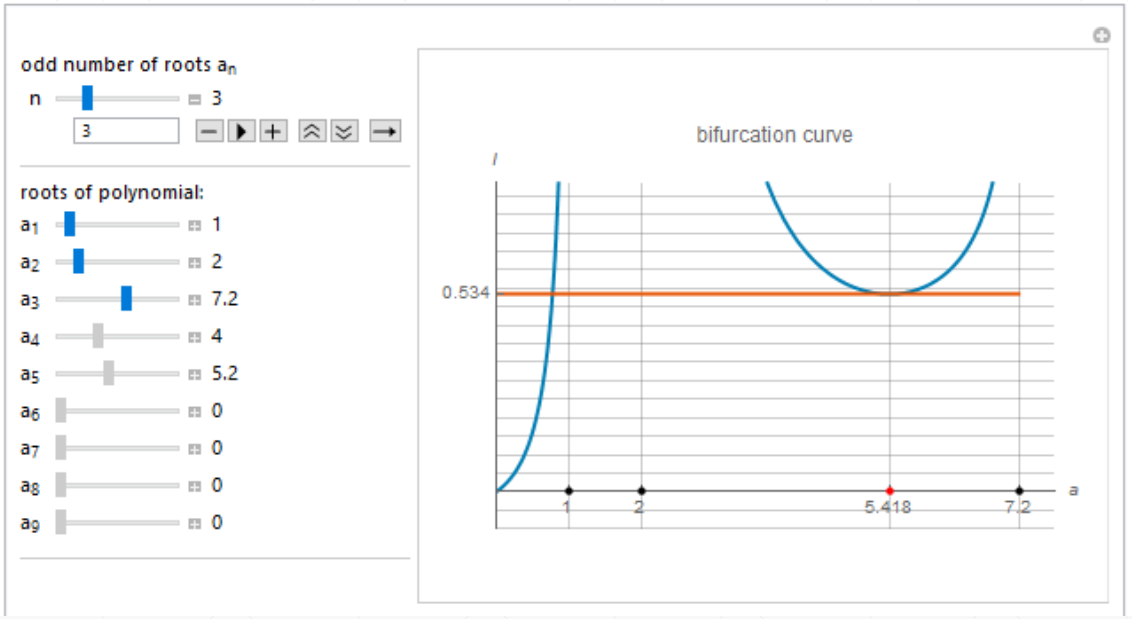
- Manipulate command = evaluation part:
  - Initialize function $\lambda(a)$ of numeric parameter a (**NumericQ**);
  - Input number of roots of polynomial $f(y)$ and exact values of roots;
  - Sort roots in ascending order (**Sort**) and find a root with maximal value (**Max**);
  - Create a polynomial function $f(y)$;
  - Create an integral function $F(y)$;
  - Find the turning points of $\lambda(a)$ using **NMinimize** and get the arrays of a and $\lambda(a)$ using **Table**;
  - Plot bifurcation curve $\lambda(a)$ in coordinate system $(a;\lambda)$ with an array of $\lambda(a)$ using **Plot**;
  - In **Epilog** parameter of **Plot** register roots of polynomial a_i (black points), turning points a (red points) and bifurcation parameters λ (ticks on λ -axis).

WDP sample using

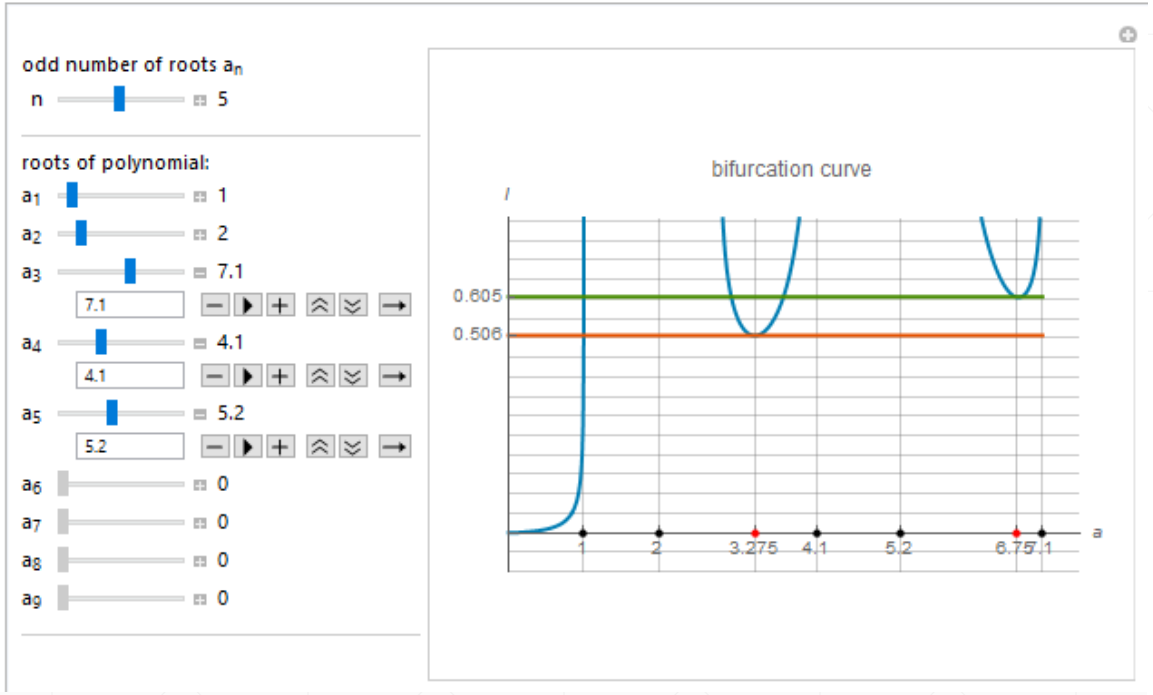


$$f(y) = (y-1)(y-2)(y-4)(y-5)(7-y)$$

$$f(y) = (y-1)(y-2)(7.2-y)$$

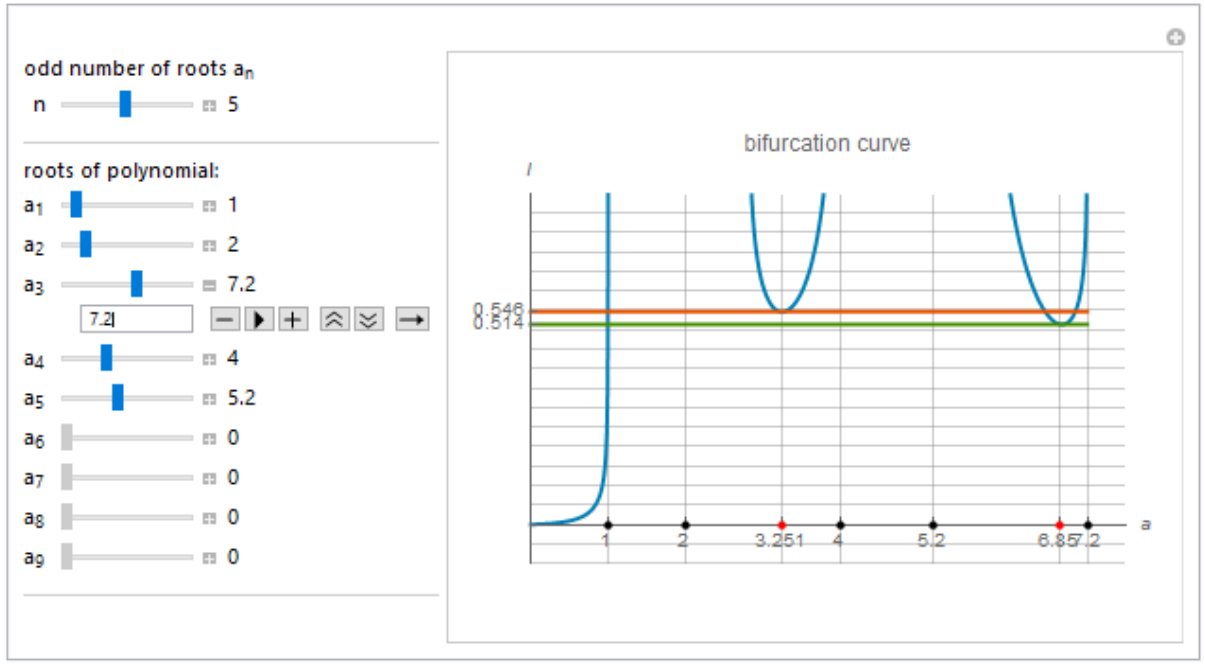


WDP sample using



$$f(y) = (y-1)(y-2)(y-4.1)(y-5.2)(7.1-y)$$

$$f(y) = (y-1)(y-2)(y-4)(y-5.2)(7.2-y)$$



Conclusion

$$f(y) = (y - a_1)(y - a_2) \dots (y - a_{2n-2})(a_{2n-1} - y)$$

- **Maximal number of positive solutions depends on the degree of polynomial and equals λ_n,** so there is one solution to the problem where $\lambda < \lambda_0$, there are two solutions where $\lambda = \lambda_0$, there are three solutions where $\lambda_0 < \lambda < \lambda_1$, etc.
- With the help of WDP we have a tool for further analysis of behavior and interconnection between the number of roots of $f(y)$, number of positive solutions of BVP and bifurcation parameters.

References

- [1] D.V. Chemkaeva, A.V. Flegontov Bifurcation diagrams for polynomial nonlinear ODE. International Conference Polynomial Computer Algebra; St. Petersburg, April 18-21 2018 / Euler International Mathematical Institute, Ed. by N.N.Vassiliev, VVM Pubishing, 2018.
- [2] P. Korman, Y. Li, T. Ouyang Computing the location and the direction of bifurcation, Math. Research Letters, 2005.
- [3] D.V. Chemkaeva Bifurcation Analysis of Some Classes of Nonlinear Boundary Value Problems with Parameter (russian), Differential Equations and Control Processes / Mathematics and Mechanics Faculty of Saint-Petersburg State University, 2018.
- [4] Wolfram Demonstrations Project https://demonstrations.wolfram.com/, 2019.