Litvinov-Maslov Dequantization of Matrix Algebras: New Insights and Techniques

Nikolayev Dmitry

Abstract. Tropical mathematics studies semifields with idempotent addition obtained via extreme logarithmic deformation of the real semifield known as Litvinov-Maslov dequantization. We investigate unobvious aspects of its generalization to the matrix case giving raise to a new class of tropical algebras that we refer to as uncanonical. We classify tropical matrix algebras obtained by dequatization of the real one and provide computational examples.

Introduction

Tropical mathematics studies semirings and semifields with idempotent addition. For example, the semifield $\mathbb{R}_{\max,+}$ is defined as the set $\mathbb{R} \cup \{-\infty\}$ equipped with addition $a \oplus b = \max(a, b)$ and multiplication $a \otimes b = a + b$, zero $\mathbb{0} = -\infty$ and unit $\mathbb{1} = 0$. The relationship between usual and tropical algebras was described by G.L. Litvinov and V.P. Maslov in terms of logarithmic deformation of the real semifield (quantization) parametrized by some paramter $h \in \mathbb{R}$ and the subsequent taking the limit when $h \to 0$ (dequantization). Other tropical algebras like $\mathbb{R}_{\max,\times}$, $\mathbb{R}_{\min,+}$ and $\mathbb{R}_{\min,\times}$ could be obtained via the same procedure depending on how exactly the original algebra is being deformed and how the limit is being taken [1].

A semimodule over a semifield is a generalization of the classical notion of a linear space over a field, wherein the corresponding scalars are the elements of a given semiring and a multiplication is defined of the ring and elements of the module [2, 3]. The matrix semimodule could be turned into matrix algebra considering only square matrices semimodule with respect to matrix multiplication. The tropical matrix algebra has two important but not equivalent constructions through Litvinov-Maslov dequationtization of real matrix. The most remarkable fact is that one of them admits one additional way of dequantization giving raise to a new classes of tropical matrix algebras that we refer to as uncanonical. We classify tropical matrix algebras obtained by dequatization of the real one and provide computational examples.

1. Dequantization of Scalar Algebras

A semifield $(\mathbb{S}, \oplus, \odot, \mathbb{0}, \mathbb{1})$ is a set $\mathbb{S}$ equipped with addition $\oplus$ and multiplication $\otimes$ operations, zero $\mathbb{0}$ and unit $\mathbb{1}$ elements such that $(\mathbb{S}, \oplus, \mathbb{0})$ is a commutative monoid, $(\mathbb{S}/\{\mathbb{0}\}, \otimes, \mathbb{1})$ is a comutative group, $\otimes$ is distributive over $\oplus$, $\mathbb{0}$ is an absorbing element $a \otimes \mathbb{0} = \mathbb{0}$. Tropical mathematics studies semifields with idempotent addition $a \oplus a = a$. Taking into account the multiplication group properties of semifilds division $a \oslash b = ab^{-1}$ is also available.

Algebra	$\mathbb{S}$	$a \oplus b$	$a \otimes b$
$\mathbb{R}_{+_h,\times}$	$\mathbb{R}_+ \cup \{h\ln(0)\}$	$h\ln(e^{\frac{a}{h}} + e^{\frac{b}{h}})$	$a \times b$
$\mathbb{R}_{+_h,\times_h}$	$\mathbb{R} \cup \{h\ln(0)\}$	$h\ln(e^{\frac{a}{h}} + e^{\frac{b}{h}})$	$a + b$

TABLE 1. Quantized scalar semirings

For example, $\mathbb{R}_{\max,+}$ is the set $\mathbb{R} \cup \{-\infty\}$ equipped with addition $a \oplus b = \max(a, b)$ and multiplication $a \otimes b = a+b$, zero $\mathbb{0} = -\infty$ and unit $\mathbb{1} = 0$. According to Litvinov-Maslov approach an explicit construction of tropical algebras listed in Table 2 is obtained via the composite map $\ell_h \circ \mu_h : \mathbb{R}_+ \to \mathbb{S}_h \to \mathbb{S}$ that sequentially transforms the real semifield $\mathbb{R}_+$ into a quantized semiring $\mathbb{S}_h$ by $\mu_h : \mathbb{R}_+ \to \mathbb{S}_h$ and then $\mathbb{S}_h$ into tropical semifields $\mathbb{S}$ by raking the corresponding limit $\ell_h : \mathbb{S}_h \to \mathbb{S}$.

Algebra	$\mathbb{S}$	$a \oplus b$	$a \otimes b$
$\mathbb{R}_{\min,\times}$	$\mathbb{R}_+ \cup \{+\infty\}$	$\min(a, b)$	$a \times b$
$\mathbb{R}_{\max,\times}$	$\mathbb{R}_+ \cup \{-\infty\}$	$\max(a, b)$	$a \times b$
$\mathbb{R}_{\min,+}$	$\mathbb{R} \cup \{+\infty\}$	$\min(a, b)$	$a + b$
$\mathbb{R}_{\max,+}$	$\mathbb{R} \cup \{-\infty\}$	$\max(a, b)$	$a + b$

TABLE 2. Tropical semirings

The role of $\mathbb{S}_h$ could be played by partially quantized $\mathbb{R}_{+_h,\times}$ or fully quantized $\mathbb{R}_{+_h,\times_h}$ semifield, where $+_h$ and $\times_h$ denote new operations induced by the variable change $x \to h\ln x$ defined as $a +_h b = \ln(\exp(a/h) + \exp(b/h))$ and $a \times_h b = \ln(\exp(a/h) \times \exp(b/h)) = a + b$ and parametrized by $h \in \mathbb{R}$ playing the role of Plank's constant. This parameter $h \in \mathbb{R}$ generates an ordered sequence of semirings $\mathbb{S}_h$ that has a limit depending on its sign defined in the following way

$$\mathbb{R}_{+_h,\times} = \begin{cases} \mathbb{R}_{\min,\times}, & h \to -0; \\ \mathbb{R}_{\max,\times}, & h \to +0; \end{cases} \qquad \mathbb{R}_{+_h,\times_h} = \begin{cases} \mathbb{R}_{\min,+}, & h \to -0; \\ \mathbb{R}_{\max,+}, & h \to +0. \end{cases}$$

2. Dequantization of Matrix Algebras

A semimodule over a semifield is a generalization of the notion of vector space over a field wherein the corresponding scalars are the elements of an arbitrary given semiring and a multiplication is defined between elements of the semiring and elements of the semimodule. Matrix semimodule could be turned into an algebra by considering square matrices with respect to their multiplication. The tropical matrix algebra has two important but not equivalent constructions through Litvinov-Maslov dequationtization of real matrix. The most remarkable fact is that one of them admits one additional way of dequantization giving raise to a new classes of tropical matrix algebras that we refer to as uncanonical.

Algebra	$\mathbb{S}_h$	$a \oplus b$	$a \otimes b$
$\mathbb{R}_h^{\langle n\times n\rangle}$	$\left(\mathbb{R} \cup \{h\ln(0)\}\right)^{n\times n}$	$A_{ij} +_h B_{ij}$	$\sum_{k=1}^n A_{ik} \times B_{kj}$
$\mathbb{R}_h^{[n\times n]}$	$\left(\mathbb{R}_+ \cup \{h\ln(0)\}\right)^{n\times n}$	$A_{ij} +_h B_{ij}$	$\sum_{k=1}^n A_{ik} \times_h B_{kj}$
$\mathbb{R}_h^{(n\times n)}$	$\left(\mathbb{R} \cup \{h\ln(0)\}\right)^{n\times n}$	$A_{ij} +_h B_{ij}$	$\sum_{hk=1}^{n} A_{ik} \times_h B_{kj}$

TABLE 3. Quantized matrix semirings

Square matrix algebra over $\mathbb{S}$ is the set $\mathbb{S}^{n\times n}$ with respect to addition $\boxplus$ and multiplication $\boxtimes$, zero O and unit I defined for $A, B \in \mathbb{S}^{n\times n}$ by the formulas

$$\{A \boxplus B\}_{ij} = A_{ij} \otimes B_{ij}, \quad \{A \boxtimes B\}_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}.$$

In our notation matrix operations are denoted by box signs and scalar operations – by circle signs. So, the tropical matrix algebra $\mathbb{S}$ has two important but non-equivalent constructions: (1) quantization-dequantization of real scalars, and the subsequent modularization of tropical scalars, (2) modularization of real scalars,and the subsequent quantization-dequantization of real matrices. As in previous case the limits coulb be taken.

Algebra	$\mathbb{S}$	$\{A \oplus B\}_{ij}$	$\{A \otimes B\}_{ij}$
$\mathbb{R}_{\min}^{\langle n\times n\rangle}$	$\left(\mathbb{R} \cup \{+\infty\}\right)^{n\times n}$	$\min(A_{ij}, B_{ij})$	$\sum_{k=1}^n A_{ik} \times B_{kj}$
$\mathbb{R}_{\max}^{\langle n\times n\rangle}$	$\left(\mathbb{R} \cup \{-\infty\}\right)^{n\times n}$	$\max(A_{ij}, B_{ij})$	$\sum_{k=1}^n A_{ik} \times B_{kj}$
$\mathbb{R}_{\min}^{[n\times n]}$	$\left(\mathbb{R} \cup \{+\infty\}\right)^{n\times n}$	$\min(A_{ij}, B_{ij})$	$\min_{k=1}^n A_{ik} \times B_{kj}$
$\mathbb{R}_{\max}^{[n\times n]}$	$\left(\mathbb{R} \cup \{-\infty\}\right)^{n\times n}$	$\max(A_{ij}, B_{ij})$	$\max_{k=1}^n A_{ik} \times B_{kj}$
$\mathbb{R}_{\min}^{(n\times n)}$	$\left(\mathbb{R} \cup \{+\infty\}\right)^{n\times n}$	$\min(A_{ij}, B_{ij})$	$\min_{k=1}^n A_{ik} + B_{kj}$
$\mathbb{R}_{\max}^{(n\times n)}$	$\left(\mathbb{R} \cup \{-\infty\}\right)^{n\times n}$	$\max(A_{ij}, B_{ij})$	$\max_{k=1}^n A_{ik} + B_{kj}$

References

[1] Litvinov G. L., Maslov dequantization, idempotent and tropical mathematics: a brief introduction, Journal of Mathematical Sciences, 140 (3), 2007.

[2] Baccelli F., Cohen G., Olsder G., J.-P. Quadrat synchronization and linearity: an algebra for discrete event systems. – New York: John Wiley & Sons Ltd, 1997.

[3] Kolokoltsov V.N., Maslov V.P. Idempotent analysis and its applications. Dordrecht, Kluwer Academic Publishers Group, 1997.

Nikolayev Dmitry
Statistical Modelling Department
Saint-Petersburg State University
Saint-Petersburg, Russia
e-mail: `NikolayevDmitry@yandex.ru`