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How much non-classicality does the Hilbert space of N-dimensional quantum system contain?

Vahagn Abgaryan, Arsen Khvedelidze, Ilya Rogojin and Astghik Torosyan

One of manifestations of incompatibility of classical and quantum descriptions of reality is the existence of negative values of the Wigner function (WF), which serves as a quantum analogue of the probability distribution function. Though, strictly speaking, the physical reflection of this deviant behaviour is unknown, it is nevertheless well established that various non-classical physical effects ranging from quantum entanglement to squeezing may accompany the WF negativity [1, 2]. Moreover, it may be speculated that there is an interplay between the intensity of non-classicality of the state and various measures characterizing the degree of negativity of WF [1, 3]. It seems that there are two natural ways to quantify the degree of negativity. First one is through distance based measures describing the remoteness of the (state described by) given WF from the set of (classical states with) non-negative WFs [1, 3]. While the second by volume based measures through the volume of the negative part of WF [1].

The aim of this short report is an estimation of the total non-classicality contained in the Hilbert space of N-dimensional system which is uniformly covered with respect to the Hilbert-Schmidt measure, through a volume based measure of negativity[1]. We define the indicator of non-classicality of the given state ρ as the relative measure

$$M_\rho = \frac{1}{Vol\ (\Omega_\rho[\nu])}\, \mathbb{M}\left\{\omega \in \Omega_\rho \mid \mathrm{W}_\rho(\omega) < 0\right\}, \tag{1}$$

while the global indicator of non-classicality as

$$\mathcal{P} = \frac{1}{Vol\,(State\ space)} \int M_\rho \, \mathrm{d}V(\rho). \tag{2}$$

Here ν is a set of parameters fixing the Stratonovich-Weyl mapping kernel [4, 5]. The estimation of $\mathcal{P}$ for infinite-dimensional systems is going to be presented in the report for the most convenient choices of the Stratonovich-Weyl kernel.

[1]Under assumption of universality of each of two kinds of measures, this, among other things, will describe the degree of remoteness of classical and quantum phase spaces.

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Quantifying non-classicality of the Hilbert-Schmidt ensemble of qubits and qutrits by Wigner functions

Vahagn Abgaryan, Arsen Khvedelidze and Astghik Torosyan

According to modern views, the Wigner quasiprobability distribution, or simply the Wigner function (WF) provides a qualitative information on many quantum phenomena occurring in diverse physical systems [1].

The WF has all the properties of statistical distributions except one: taking negative values for some quantum states, the WF turns to be not a proper distribution and hence indicates the existence of truly quantum features which cannot be described within the classical statistical paradigm. Deviation of the Wigner quasiprobability distribution from a proper statistical distribution of a physical system is interpreted as an evidence of non-classicality, or quantumness. In our report, based on the recently elaborated method of construction of the WF of a finite dimensional system [2, 3], we will discuss the following measures/indicators for quantification of non-classicality of a finite-dimensional system:

1. **The negativity probability** defined for an arbitrary ensemble of a random quantum state as the ratio:

$$\mathcal{P}^{(-)}(N) := \frac{Number\ of\ states\ with\ negative\ WF}{Total\ number\ of\ generated\ states}\,.$$

2. **KZ-indicator** introduced by A.Kenfack and K.Zyczkowski [4]:

$$\delta(\varrho) = \int_{\Omega_N} \mathrm{d}\Omega_N \big|W_\varrho(\Omega_N)\big| - 1\,. \tag{1}$$

   Here the notation $\big|W\big|$ means the absolute value (modulus) of the Wigner function $W(\Omega_N)$, defined on the phase-space manifold Ω_N.

3. **Global indicator of non-classicality** defined as the following quantity:

$$\mathcal{Q} = \frac{\text{Volume of orbit space}\ \ \mathcal{O}[\mathfrak{P}_N^{(+)}]}{\text{Volume of total orbit space}\ \ \mathcal{O}[\mathfrak{P}_N]}\,, \tag{2}$$

   where $\mathfrak{P}_N^{(+)}$ denotes the subspace of a state space $\mathfrak{P}$ with non-negative WF:

$$\mathfrak{P}_N^{+} = \{\varrho \in \mathfrak{P}_N \ \mid W_\varrho(\Omega_N) \geq 0\,,\ \forall\ \Omega_N\}\,. \tag{3}$$

It is assumed that the volume in (2) is calculated with respect to a Riemannian metric induced by mapping of a state space $\mathfrak{P}_N$ to the orbit space $\mathcal{O}[\mathfrak{P}_N]$:

$$\mathfrak{P}_N \longrightarrow \mathcal{O}[\mathfrak{P}_N] := \frac{\mathfrak{P}_N}{SU(N)}\,. \tag{4}$$

All the above mentioned non-classicality measures will be exemplified by considering the Hilbert-Schmidt ensemble of qubits and qutrits.

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Galois elliptic function and its symmetries

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Abstract. The vast subject of elliptic integrals, functions and curves has numerous applications in physics. Two distinct approaches to investigating elliptic functions have become "classical": that of Carl Jacobi and that of Karl Weierstrass. Two distinct chapters are dedicated to these two approaches in Whittaker and Watson (famous) "Course of Modern Analysis" without attempts of unification. Some have thus claimed that the Weierstrass approach is more suitable for "theoretical" research, whereas the Jacobi elliptic functions arise "more frequently" in applications. Yet and indeed such dichotomy is artificial and the study of elliptic functions and curves can and must be naturally united via an algebraic approach, readily producing a most canonical "essential" elliptic function which preserving transformations acquire "simplest" forms. Although such "natural" building block, to which we (justifiably) ought to refer as *the Galois elliptic function*, has been only recently introduced, its exploitation has already been quite fruitful. Not merely for efficiently regenerating known and established results but for attaining new calculations which previously seemed too cumbersome to pursue via either the Jacobi or the Weierstrass approach. Here, at PCA 2019, we aim at demonstrating the methodological significance of this naturally algebraic approach via applying it to a few but quite fundamental problems of classical mechanics, thereby producing formidable, hardly standard and highly efficient solutions!

1. Preliminary definitions

For each nonzero c, define a *homothety operator* $H(c)$ via its action on an arbitrary function f:

$$H(c)f(x) := cf\left(x/c\right).$$

Thus, the graph of the function f when subjected to homothety, with ratio c, with the origin being the fixed point, yields the graph of the function $H(c)f$.
We shall use an upper subscript to indicate the functional transformation

$$f^c(x) := cf\left(\sqrt{c}x\right),$$

which we shall not hesitate to explicitly rewrite whenever a risk of confusing such a notation, with the common use of upper subscripts for denoting powers, arises. While keeping in mind that the square root is not single valued, we shall assume, unless we indicate otherwise, its values to lie in the right half plane without the imaginary negative ray. With this choice of branch for the square root one guarantees that

$$\sqrt{\overline{x}} = \overline{\sqrt{x}},$$

with the bar denoting complex conjugation.
Let $\mathcal{H}$ denote the upper half of the complex plane

$$\mathcal{H} := \{z : \ \mathrm{Im}(z) > 0\}.$$

The group $PSL(2,\mathbb{R})$ acts faithfully on $\mathcal{H}$, and constitutes the group of its conformal automorphisms.
Let $\mathcal{D}$ denote the fundamental domain for the action of the modular group $PSL(2,\mathbb{Z})$, being a discrete subgroup of $PSL(2,\mathbb{R})$, upon $\mathcal{H}$

$$\mathcal{D} := \mathcal{H}/PSL(2,\mathbb{Z}).$$

Denoting by $\mathcal{L}$ the set of lattices, that is the set of discrete subgroups of rank 2, in $\mathbb{C}$, we may identify $\mathcal{D}$ with the quotient $\mathcal{L}/\mathbb{C}^{\times}$, that is, the quotient of the action of the multiplicative group of invertible elements of $\mathbb{C}$ upon $\mathcal{L}$.
The Eisenstein series of index k is a function G_k, on the upper half plane $\mathcal{H}$, determined by the equality

$$G_k(x) := \sum_{(n,m)\in\mathbb{Z}\oplus\mathbb{Z}\setminus(0,0)} (m\,x+n)^{-2k}.$$

When the index k is an integer strictly exceeding 1, G_k is a modular form of weight k. In other words, it is then a holomorphic function on the extended upper half plane $\mathcal{H}\cup\infty$, satisfying the relation

$$G_k\left(\frac{ax+b}{cx+d}\right) = (cx+d)^{2k}G_k(x) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Exploiting the identity

$$G_k(\infty) = 2\zeta(2k),$$

where ζ is the Riemann ζ-function and setting

$$\Delta := 4h_2^3 - 27h_3^2, \quad h_2 = 15G_2,\ h_3 = 35G_3,$$

one finds that

$$\Delta(\infty) = 4\left(\frac{\pi^4}{3}\right)^3 - 27\left(\frac{2\pi^6}{27}\right)^2 = 0.$$

Thereby Δ is a cusp form of weight 6.
The Klein j-invariant is defined as

$$j := \frac{4h_2^3}{\Delta} \tag{1}$$

and can be viewed, being a modular function of weight 0, as a bijection from the extended fundamental domain $\mathcal{D} \cup \infty$ onto the Riemann sphere $\mathbb{C} \cup \infty$ with a (simple) pole at infinity. The j-invariant is evidently fixed under the transformation

$$(h_2, h_3) \mapsto (c^2 h_2, c^3 h_3).$$

For an arbitrary pair (h_2, h_3), subject to the condition that Δ does not vanish, *the Weierstrass elliptic function* $\wp$ is the solution of the differential equation

$$y'^2 = 4\left(y^3 - h_2 y - h_3\right) = 4(y - e_1)(y - e_2)(y - e_3) \tag{2}$$

with a (double) pole at zero. The function $\wp$ represents a one parameter family of Weierstrass elliptic functions

$$\mathcal{F}_\wp := \{\wp^c : \ x \mapsto c\,\wp(\sqrt{c}\,x), \ c \in \mathbb{C}^\times\}. \tag{3}$$

Each $\wp^c \in \mathcal{F}_\wp$ satisfies a corresponding differential equation

$$y'^2 = 4\left(y^3 - c^2 h_2 y - c^3 h_3\right) = 4(y - ce_1)(y - ce_2)(y - ce_3), \tag{4}$$

and the whole family $\mathcal{F}_\wp$ shares a "common" j-invariant $j(\tau)$, corresponding to a single point τ in $\mathcal{D}$.

For a fixed k, we define *the Jacobi elliptic function* sn as the vanishing at zero solution of the differential equation

$$y'^2 = \left(1 - y^2\right)\left(1 - k^2 y^2\right),$$

whose leading Maclaurin series coefficient is 1. The constant k is called *the elliptic modulus*.

Every Jacobi elliptic function sn represents a one parameter family of elliptic functions

$$\mathcal{F}_{\mathrm{sn}} := \{H\left(\sqrt{c}\,\right)\mathrm{sn} : \ c \in \mathbb{C}^\times\}. \tag{5}$$

Each $H\left(\sqrt{c}\,\right)\mathrm{sn} \in \mathcal{F}_{\mathrm{sn}}$ satisfies a corresponding differential equation

$$y'^2 = \left(1 - y^2/c\right)\left(1 - k^2 y^2/c\right).$$

2. The Galois essential elliptic function

Let α be a parameter, $\alpha \in \mathbb{C} \setminus \{-2/3, 2/3\}$. Define an *essential elliptic function* $\mathcal{R}_\alpha$ as the solution of the differential equation

$$y'^2 = 4\,y\left(y^2 + 3\alpha y + 1\right), \tag{6}$$

with a (double) pole at zero.

An essential elliptic function $\mathcal{R}_\alpha$ differs from Weierstrass elliptic function $\wp_\alpha$ by an additive constant. Explicitly, if $\wp_\alpha$ is the Weierstrass elliptic function, satisfying the differential equation

$$y'^2 = 4\left(y^3 - (3\alpha^2 - 1)\,y - \alpha(1 - 2\alpha^2)\right) = 4(y - \alpha)\left(y - (\alpha - \beta)\right)\left(y - (\alpha - 1/\beta)\right), \qquad \beta := (3\alpha + d)/2, \ \ d^2 := 9\alpha^2 - 4, \tag{7}$$

then

$$\mathcal{R}_\alpha = \wp_\alpha - \alpha.$$

In particular, the essential elliptic function $\mathcal{R}_\alpha$ coincides with the Weierstrass function $\wp_\alpha$ when $\alpha = 0$. Incidentally, the discriminant of either the cubic appearing on the right hand side of equation (6) or the cubic in (7), which we shall call *the discriminant associated with* α, is $d^2 = (\beta - 1/\beta)^2 = 9\alpha^2 - 4$, so it does not vanish since, by assumption, $\alpha \neq \pm 2/3$.
Each Weierstrass elliptic function $\wp$, satisfying a differential equation (2), represents a family of Weierstrass elliptic functions $\mathcal{F}_\wp$ (3). Fix such a function $\wp$, representing such a family $\mathcal{F}_\wp$, so as to assume the pair (h_2, h_3) and its corresponding discriminant Δ being fixed. Let τ be the unique point in $\mathcal{D}$ whose image $j = j(\tau)$, given by (1), matches the j-invariant of this family. We shall associate six essential elliptic functions with τ, and thus with $\mathcal{F}_\wp$, as long as τ is neither a cube root of 1 nor a square root of -1.
If c is a root of the hexic

$$c^6 - \frac{3h_2}{\Delta}\, c^2 + \frac{1}{\Delta} \tag{8}$$

then the differential equation (4), satisfied by $\wp^c$ for that particular choice of c, acquires the form of the differential equation (7) satisfied by $\wp_\alpha$, where α is determined by the choice of c. The assumption $\tau \neq i$ implies that $h_3 \neq 0$ and the hexic (8), then being separable, possesses six distinct roots, which we might enlist in three pairs $(c_1, -c_1)$, $(c_2, -c_2)$ and $(c_3, -c_3)$, where

$$c_1^2 = \frac{1}{(e_1 - e_2)(e_1 - e_3)},\ c_2^2 = \frac{1}{(e_2 - e_3)(e_2 - e_1)},\ c_3^2 = \frac{1}{(e_3 - e_1)(e_3 - e_2)}.$$

For each root pair $(c_n, -c_n)$, $n = 1, 2, 3$, we obtain a corresponding factorization of the cubic on the right hand side of (4), with c replaced by c_n and $-c_n$, respectively

$$\begin{cases} y^3 - c_n^2 h_2 y - c_n^3 h_3 = (y - \alpha_n)\,(y - (\alpha_n - \beta_n))\,(y - (\alpha_n - 1/\beta_n)) \\ \\ y^3 - c_n^2 h_2 y + c_n^3 h_3 = (y + \alpha_n)\,(y + (\alpha_n - \beta_n))\,(y + (\alpha_n - 1/\beta_n)) \end{cases}$$

Here, the additional assumption $\tau \neq e^{2\pi i/3}$ guarantees that the three values α_1, α_2 and α_3 are pairwise distinct, whereas we have already ensured, with the assumption $\tau \neq i$, that none of the them vanishes. Thus, six distinct Weierstrass functions $\wp_{\alpha_n}$, $\wp_{-\alpha_n}$, $n = 1, 2, 3$, are obtained, and we may, as promised, associate six essential elliptic functions $\mathcal{R}_{\alpha_n}$, $\mathcal{R}_{-\alpha_n}$, $n = 1, 2, 3$, with $\mathcal{F}_\wp$. Plainly, for each n, $n = 1, 2, 3$, we have

$$\mathcal{R}_{\alpha_n} = \wp_{\alpha_n} - \alpha_n,\ \mathcal{R}_{-\alpha_n} = \wp_{-\alpha_n} + \alpha_n, \quad \alpha_n = c_n e_n,$$

and for each n, $n = 1, 2, 3$, the discriminant associated with α_n is

$$d_n^2 = 9\alpha_n^2 - 4 = c_n^6 \Delta.$$

The three pairs $(d_1, -d_1)$, $(d_2, -d_2)$ and $(d_3, -d_3)$ can be viewed as the roots of the hexic

$$4\left(d^2 + 1\right)^3 - 27jd^2 = 4\left(d^6 + 3d^4 + 3\left(1 - 9j/4\right)d^2 + 1\right), \tag{9}$$

which is separable aside from the two special cases corresponding to $\tau = i$ and $\tau = e^{2\pi i/3}$. We now consider these two cases.

If $\tau = i$ then $h_3 = 0$, $j = 1$ and the hexic (9) factors to

$$4\left(d^2+4\right)\left(d^2-1/2\right)^2.$$

If $\tau = e^{2\pi i/3}$ then $h_2 = 0$, $j = 0$ and the hexic (9) factors to

$$4\left(d^2+1\right)^3.$$

Instead of writing the hexic (9) we could have written a hexic which roots are the three pairs $(\alpha_1, -\alpha_1)$, $(\alpha_2, -\alpha_2)$ and $(\alpha_3, -\alpha_3)$. This is the hexic

$$4\left(3\alpha^2-1\right)^3 - j\left(9\alpha^2-4\right). \tag{10}$$

Only three distinct roots and three essential elliptic functions correspond to $\tau = i$. These are

$$\left\{\mathcal{R}_\alpha :\ \alpha = 0,\ \pm\sqrt{2}/2\right\}.$$

Only two distinct roots and two essential elliptic functions correspond to $\tau = e^{2\pi i/3}$. These are

$$\left\{\mathcal{R}_\alpha :\ \alpha = \pm\sqrt{3}/3\right\}.$$

The hexic polynomials (8), (9) and (10), when viewed as cubic polynomials of c^2, d^2 and α^2, respectively, correspond to one and the same point in $\mathcal{D}$. In other words, theses three cubics possess the same j-invariant. The latter is obtained from the j-invariant of the original cubic (2) via the transformation

$$j \mapsto \frac{j}{j-1}$$

which constitutes a linear fractional transformation of order two, with zero being its only fixed point. Note that the discriminant of the cubic (8) for c^2 does not vanish when $\tau = e^{2\pi i/3}$, although it does for either one of the cubic polynomials (9) for d^2 or (10) for α^2.

We may "restore" the fundamental domain by adding two functions, corresponding to the two excluded values $\pm 2/3$ for α. When $\alpha = \pm 2/3$ equation (6) degenerates to

$$y'^2 = 4\,y\,(y \pm 1)^2,$$

and we, thus, regard the functions ctg^2 and cth^2 as the functions corresponding to $\alpha = 2/3$ and $\alpha = -2/3$, respectively, where

$$\mathrm{ctg}^2(x) = -\left(\frac{e^{2ix}+1}{e^{2ix}-1}\right)^2, \quad \mathrm{cth}^2(x) = \left(\frac{e^{2x}+1}{e^{2x}-1}\right)^2,$$

and we regard the functions csc^2 and csh^2 as the functions corresponding to $\alpha = +\infty$ and $\alpha = -\infty$, respectively, where

$$\mathrm{csc}^2(x) = -\left(\frac{2}{e^{ix}-e^{-ix}}\right)^2, \quad \mathrm{csh}^2(x) = \left(\frac{2}{e^{x}-e^{-x}}\right)^2.$$

3. The Galois alternative elliptic function

Define an *alternative elliptic function* $\mathcal{S}_\beta$ as the vanishing at zero solution of the differential equation

$$y'^2 = \left(1 - \beta y^2\right)\left(1 - y^2/\beta\right),$$

whose leading Maclaurin series coefficient is 1.
Set the elliptic modulus k equal to β. The alternative elliptic function is then a representative of the family $\mathcal{F}_{\mathrm{sn}}$ introduced in (5), namely

$$\mathcal{S}_\beta = H\left(\sqrt{\beta}\,\right)\mathrm{sn},$$

and its square is the reciprocal of an essential elliptic function, namely

$$\mathcal{S}_\beta^2 = 1/\mathcal{R}_{-\alpha}. \tag{11}$$

The degenerate elliptic function $\mathcal{S}_{-1}$ and $\mathcal{S}_1$ correspond to $\alpha = 2/3$ and $\alpha = -2/3$, respectively

$$\mathcal{S}_{-1}(x) := \tan(x) = \frac{e^{2ix} - 1}{i\left(e^{2ix} + 1\right)}, \quad \mathcal{S}_1(x) := \mathrm{tgh}(x) = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

Two functions are candidates for representing the degenerate elliptic function $\mathcal{S}_0$, corresponding to $\alpha = +\infty$ and $\alpha = -\infty$, respectively

$$\mathcal{S}_+(x) := \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \quad \mathcal{S}_-(x) := \mathrm{snh}(x) = \frac{e^{x} - e^{-x}}{2}.$$

4. Essential relations amongst Galois elliptic functions

With

$$k^2 = \frac{3\alpha - 2}{3\alpha + 2} = \left(\frac{\beta - 1}{\beta + 1}\right)^2,$$

the function

$$\mathcal{S} := H\left(\sqrt{\frac{1}{d}}\,\right)\mathcal{S}_k = H\left(\sqrt{\frac{k}{d}}\,\right)\mathrm{sn}$$

satisfies the differential equation

$$y'^2 = \left(1 - (3\alpha + 2)\, y^2\right)\left(1 - (3\alpha - 2)\, y^2\right),$$

and, moreover, the identity

$$\mathcal{S}^{-2} = \mathcal{R}_\alpha + 3\alpha + \mathcal{R}_\alpha^{-1} = \mathcal{W}_\alpha + 2\alpha, \tag{12}$$

where the sum

$$\mathcal{W}_\alpha := \mathcal{R}_\alpha + \alpha + \mathcal{R}_\alpha^{-1}$$

is viewed as a Weierstrass function, satisfying the equation

$$y'^2 = 4\left(y + 2\alpha\right)\left(y - \alpha + 2\right)\left(y - \alpha - 2\right),$$

holds. Thus, identity (12) establishes a well-known (yet, perhaps, somewhat disguised) link between Weierstrass and Jacobi elliptic functions. The function

$\mathcal{W}_\alpha$ represents a family $\mathcal{F}_{\mathcal{W}_\alpha} = \{\mathcal{W}_\alpha^c : \ c \in \mathbb{C}^\times\}$, corresponding to a point $\tau \in \mathcal{D}$. We proceed to determining a particular representative $\mathcal{W}_\alpha^c$ differing from an essential elliptic function, associated with τ, by an additive constant. Evidently, for $c = 1/d = 1/\sqrt{9\alpha^2 - 4}$,

$$\mathcal{W}_\alpha^c + 2c\alpha = \mathcal{R}_{-2c\alpha}$$

is then satisfied by that, sought for, representative $\mathcal{W}_\alpha^c$, whence

$$\mathcal{W}_\alpha + 2\alpha = \mathcal{R}_{-2c\alpha}^d,$$

and

$$\mathcal{S}^2 = 1/\mathcal{R}_{-2c\alpha}^d,$$

reestablishing that

$$\mathcal{S}_k^2 = 1/\mathcal{R}_{-2c\alpha},$$

in agreement with formula (11).
Applying the identity

$$\frac{1}{4}\left(\frac{\mathcal{R}'_\alpha}{\mathcal{R}_\alpha}\right)^2 = \mathcal{R}_\alpha + 3\alpha + \mathcal{R}_\alpha^{-1},$$

to formula (12), we arrive at the equalities

$$\mathcal{S}^{-2} = \frac{1}{4}\left(\frac{\mathcal{R}'_\alpha}{\mathcal{R}_\alpha}\right)^2 = \left(\frac{\mathcal{S}'_{-\beta}}{\mathcal{S}_{-\beta}}\right)^2,$$

implying the relation

$$\int_{x_0}^{x} \mathcal{S}^{-1}(x)\, dx = \ln\left(\mathcal{S}_{-\beta}\left(x\right)\right),$$

where $2x_0$ is the root of $\mathcal{R}_\alpha$ closest to zero. Globally, of course, the latter equality holds true "modulo $2\pi i$".

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The symbolic problems associated with Runge-Kutta method

Edik Ayryan, Mikhail Malykh, Leonid Sevastianov and Yu Ying

Abstract. The talk is devoted to application of the computer algebra system Sage to search of coefficients of Butcher matrix and the organization of calculations by implicit finite difference schemes.

The Runge-Kutta method is the most popular numerical method for solving of ordinary differential equations (ODE), however the development of this method indicate some symbolic problems.

Let's review some results on Runge-Kutta scheme [1]. For an autonomous system

$$\dot{\vec{x}} = \vec{F}(\vec{x}) \tag{1}$$

Runge-Kutta scheme with s stages can be written as

$$\vec{k}_i = \vec{F}\left(\vec{x} + dt\sum_{j=1}^{s} a_{ij}\vec{k}_j\right), \quad i = 1, 2, \ldots, s \tag{2}$$

and

$$\hat{\vec{x}} = \vec{x} + dt\sum_{i=1}^{s} b_i\vec{k}_i. \tag{3}$$

We will write the coefficients a_{ij} and b_i in Butcher matrix, for ex. for $s = 2$

c_1	a_{11}	a_{12}
c_2	a_{21}	a_{22}
	b_1	b_2

Hereinafter

$$c_i = \sum_{j=1}^{s} a_{ij}.$$

These coefficients select so that the difference scheme approximate the ode (1) with some order p. For $s = p = 4$ appropriate numerical values for the coefficients was found by Kutta in the XIX century. The usage of Runge-Kutta scheme for

the solving of given ODEs means the numerical calculations in the floating-point arithmetic.

However the problem about the finding the coefficients of Butcher matrix is pure algebraic, so now we can try to solve it with the help of computer algebra systems like Sage or Maple. This is the first symbolic problems associated with Runge-Kutta method. In our paper we want to present our algorithms for symbolical calculation of the Butcher matrix and its realization in Sage.

It should be noted that conditions of approximation and other conditions doesn't define the coefficients of Butcher matrix unambiguous. From geometrical point of view the list of coefficients of Butcher matrix is a point in affine space (Butcher space) and appropriate points form a variety in this space. As in Sage there are some tools for a research of a set of solutions of systems of the algebraic equations, we can try to describe the varieties in Butcher spaces.

The Runge-Kutta scheme is explicit iff

$$a_{ij} = 0 \quad j \geq i,$$

in this case the numerical calculation don't demand the solving of nonlinear algebraic equations. This is most investigated case, for small s the Butcher varieties ware described by Butcher oneself, now we know approximate values of a_{ij} for schemes with $s = 9$ [2].

The implicit Runge-Kutta scheme is interesting because they can save some symbolic properties of exact solution. By Cooper theorem symplectic Runge-Kutta scheme save exactly all quadratic integrals of motion [3]. The symplecticity give algebraic conditions for Butcher matrix, we can try to investigate the varieties in Butcher space by the help of Sage.

In our numerical experiments these varieties were rational with singularity at point with $a_{ij} = 0$. Thus there are infinite set of Butcher matrices with rational coefficients.

Organization of calculations by the help of implicit Runge-Kutta scheme demands the solving of algebraic equations on each step. Investigation of the algebraic system is the second symbolic problem associated with Runge-Kutta scheme which we can try to solve by Sage.

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Point symmetries of Painleve VI.

M. V. Babich

Abstract. I will talk about a puzzling resembling of two different objects, namely two classes of equivalence of Painleve VI equations. One class is an orbit of the group of birational point symmetries of the Painleve VI equation. This group is the symmetric group of four objects Σ_4. The second one is an orbit of much more wide group of arbitrary point transformations of Painleve VI equation. I will demonstrate that the subgroup of all point transformations preserving the form of the Painleve VI equation is just Σ_4. The talk is based on the joint work with L. A. Bordag.

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Theorem Proving for Minimal Logic using Machine Learning Techniques

A. Baghdasaryan

Abstract. Automated theorem provers based on different systems of minimal logic experience some difficulties because of many problems. One of them is a stoup selection rule, when a formula from the context should be selected to be considered as a stoup. Neural networks are added to these systems of minimal logic and they are used to determine which formula from the context will become a stoup. This partially solves the problem of rule selection and gives reduction of time in theorem proving.

Introduction

There are different kind of systems in which rule selection problem leads to proof search inefficiency issues. Because of that problem automated theorem provers based on that systems experience some difficulties. Two systems for propositional fragment of minimal logic ($SwMin$ and $ScMin$) were introduced in [1]. In these systems the problem of rule selection remains unsolved. There is a stoup selection rule in $SwMin$, when a formula from the context should be selected to be considered as a stoup. Though this is insufficient as it requires many branches to prove, which may be unnecessary. We extend those systems to the minimal fragments of first order predicate logic $SwMinPred$ and $ScMinPred$ and prove their equivalence to Hentzen type systems considered in [2]. We developed prover $SwProv$ based on $SwMinPred$ system. To avoid rule selection problems the neural networks are deployed in $SwProv$ prover ($SwNNProv$), which helps us to make a "right" decision. At each step of the proof, if there are multiple choices of the inference rule to be applied at the current step, neural network is used to determine which formula from the context will become a stoup.

1. Sequent To Vector Transformation

Firstly all formulas in sequents are represented in Skolem standard form. To be able to use neural networks in the proof search it is necessary to train network model against provable sequents. To proceed with that we introduce numerical representation for the sequents assigning a specific number to each symbol. Based on that representation similar formulas will get identical vectors. After that autoencoder [3] is trained to get fixed length encoding for each sequent. As a result we get numerical representation for sequents.

2. Neural Networks in Proof Search

2.1. Data collection

Standard library for first-order predicate logic problems are used as a training dataset. For each element in training set $SwProv$ prover is run and which generates training examples. At each point of proof tree, where a stoup formula has to be selected , all sequents in that branch of tree are considered and sequence of vectors is generated by "Sequent to Vector" transformation. To differentiate successful outcomes while training neural network one needs to take numerical representation for each stoup candidate formula and corresponding ground truth label (whether this is the right selection).

2.2. Neural Network Architecture

Used neural network model consists of gated reclified unit(GRU) [4] as recurrent layer and 2-dense layers with skip connections [6] and residual blocks [5].

The output of recurrent layer (feature vector extractor module) is concatenated with numerical representation of stoup candidate and then is mapped to 2-length one-hot encoded vector via dense layer with softmax activation function. As a final step cross entropy is used as a loss function:

$$H_p(q) = -\frac{1}{N}\sum_{i=1}^{N} y_i * \log(p(y_i)) + (1 - y_i) * \log(1 - p(y_i)) \tag{1}$$

, where y is the label and $p(y)$ is the predicted probability of the candidate formula being right for all N examples.

Conclusion

In result of constructing new proof systems for minimal logic of predicates and deploying concept of neural network in the prover experiments revealed proof search space reduction and the level of accuracy up to 75% training 150 epochs. Compared to the prover without neural network time spent for the proof is reduced for almost twice.

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Some properties of doubly symmetric periodic solutions to Hamiltonian system

Alexander Batkhin

Abstract. We consider the structure of doubly symmetric periodic solutions to a Hamiltonian system with two degrees of freedom, which canonical equations of motion are invariant under the action of a forth order discrete group of linear automorphisms of the extended phase space. Structure and bifurcations of doubly symmetric periodic solutions are investigated. It is shown that in the case of period doubling bifurcation there always exists a pair of singly symmetric solutions with double period. Some examples of families of doubly symmetric periodic solutions of the Hill problem and of the restricted three-body problem (in the case of equal masses) are considered.

Introduction

Consider a time-independent non-integrable Hamiltonian system with two degrees of freedom, which canonical equations of motion posses the only first integral $H(\mathbf{z}) = h$, where $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathbb{M} \equiv R^4$ and $H(\mathbf{z})$ is a smooth Hamiltonian function of a system. Let the equations of motion

$$\dot{\mathbf{z}} = J \operatorname{grad} H(\mathbf{z}), \text{ here } J - \text{symplectic unit}, \tag{1}$$

are invariant under the discrete group $\mathcal{G} \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$ of linear automorphisms of the extended phase space $\mathbb{R} \times \mathbb{M}$. Two generators g_1, g_2 of the group $\mathcal{G}$ are involutive operators, i.e. $g_i^2 = \mathrm{id}$, the third non-trivial transformation $g_3 = g_1 \circ g_2 = g_2 \circ g_1$. Then each solution $\mathbf{z}(t, \mathbf{z}_0)$ to equation (1) with initial condition $\mathbf{z}(0) = \mathbf{z}_0$ belongs to one of the following group depending on the type of symmetry.

- **Non-symmetric** solutions, which change under any automorphism g_i,$i = 1, 2, 3$.
- **Singly symmetric** solutions, which are invariant under only one automorphism g_i,$i = 1, 2, 3$.
- **Doubly symmetric** solutions, which are invariant under any automorphism g_i, $i = 1, 2, 3$.

Let $\mathbf{z}(t, \mathbf{z}_0)$ be a periodic solution with period T. In the case of time-independent system solutions $\mathbf{z}(t, \mathbf{z}_0)$ belongs to a family of one-parametric periodic solutions. The parameter of such family is a value of the first integral $H(\mathbf{z})$ of system (1). The family of periodic solutions can be either closed or it has a natural termination. Such characteristics of a periodic solution as its dimension, period T, corresponding value h of the first integral, stability S change smoothly along the family, whereas the type of the symmetry is the global invariant of the family. A family can intersect another one sharing a common solution, but any family can be continued further in an unique way.

Let consider that the phase coordinates $\mathbf{z}$ are chosen in a such manner that the involutive transformation g_i has the form $g_i : (t, \mathbf{z}) \to (\sigma t, G_i\mathbf{z})$, where $\sigma = \pm 1$, G_i is a constant matrix. Let $\Sigma_i \equiv \{\mathbf{z} | g_i(\mathbf{z}) = \mathbf{z}\}$ is an invariant set of the transformation g_i. Then there exist g_i-invariant periodic solutions, which are completely defined by a part of their phase trajectories contained between points $\mathbf{z}_0$ and $\mathbf{z}(T/2)$ lying on the Σ_i.

We study in linear approximation the dynamics near doubly symmetric periodic solution $\mathbf{z}(t, \mathbf{z}_0)$ and provide its bifurcation analysis as well.

1. Properties of doubly symmetric solution

Dynamics of the system (1) in the vicinity of a solution $\mathbf{z}(t)$ is described in linear approximation by matrix $Z(t, \mathbf{z})$, which is the solution to the Cauchy problem of the Poincare variational equation

$$\dot{Z} = J\mathrm{Hess}\, H(\mathbf{z})Z, \quad Z_0 = E^4,$$

where $\mathrm{Hess}\, H(\mathbf{z})$ is the Hessian of function H computed along the solution $\mathbf{z}(t, \mathbf{z}_0)$. Matrix $Z(t)$ is symplectic: $Z^T JZ = J$ and its characteristic polynomial $P(\lambda)$ is reciprocal. Monodromy matrix M of periodic solution $\mathbf{z}(t, \mathbf{z}_0)$ is $Z(T, Z_0)$ with a property $Z(t+T) = Z(t)M$. Eigenvalues ρ_k of matrix M are called *multiplicators*. Monodromy matrix M has the following properties:

- multiplicators ρ_k are mutually complex conjugate and mutually inverse;
- matrix M has eigenvector $\mathbf{v}_1 \equiv J\,\mathrm{grad}\, H(\mathbf{z}_0)$, corresponding to multiplicator $\rho_{1,2} = 1$ with multiplicity 2;
- characteristic polynomial $P(\lambda)$ of matrix M is factorized

$$P(\lambda) = (\lambda - 1)^2 \left(\lambda^2 - 2S\lambda + 1\right),$$

where S is *stability index* of the periodic solution $\mathbf{z}(t, \mathbf{z}_0)$.

If solution $\mathbf{z}(t, \mathbf{z}_0)$ is singly g_i-symmetric with initial condition $\mathbf{z}_0 \in \Sigma_i$, then in a half of period $\mathbf{z}(T/2, z_0) \in \Sigma_i$, and monodromy matrices computed from the points $\mathbf{z}_0$ and $\mathbf{z}(T/2)$ are correspondingly

$$M_i = \widetilde{G}_i Z^{\mathrm{T}}(T/2)\widetilde{G}_i Z(T/2), \quad \widetilde{M}_i = Z(T/2)\widetilde{G}_i Z^{\mathrm{T}}(T/2)\widetilde{G}_i,$$

where $\widetilde{G}_i = G_i J$, $i = 1, 2$.

If solution $\mathbf{z}(t, \mathbf{z}_0)$ is doubly symmetric with initial condition $\mathbf{z}_0 \in \Sigma_i$, then in a quarter of the period $\mathbf{z}(T/4, \mathbf{z}_0) \in \Sigma_{3-i}$ and monodromy matrices computed from the points $\mathbf{z}_0$ and $\mathbf{z}(T/4)$ are

$$M_i = \left[\widetilde{G}_i Z^{\mathrm{T}}(T/4)\widetilde{G}_{3-i}Z(T/4)\right]^2, \quad \widetilde{M}_i = \left[Z(T/4)\widetilde{G}_i Z^{\mathrm{T}}(T/4)\widetilde{G}_{3-i}\right]^2.$$

Matrices M_i and $\widetilde{M}_i$ are similar but the similarity transformation is too awkward. The stability index of such solution can be computed by formula

$$S = 2\left(1 + 2Z_{32}Z_{14} + 2Z_{41}Z_{23} - 4Z_{11}Z_{33} + 2Z_{31}Z_{13} - 2Z_{42}Z_{24}\right)^2 - 1,$$

where Z_{ij} are components of matrix $Z(T/4, Z_0)$. So S gets its minimal value equals to 1. Special structure of monodromy matrix of symmetric solution yields to presence of inner symmetry of M:

$$m_{11} = m_{33}, m_{22} = m_{44}, m_{12} = -m_{43}, m_{14} = -m_{23}, m_{21} = -m_{34}, m_{32} = -m_{41}.$$

2. Bifurcations of periodic solutions

In [1] a linear transformation with symplectic and orthogonal matrix A was proposed which is completely defined by normalized vector $\mathcal{H}$ of the phase velocity $\mathbf{v}_1(\mathbf{z}_0)$: $\mathcal{H} = \mathbf{v}_1(\mathbf{z}_0)/|\mathbf{v}_1(\mathbf{z}_0)|$.

The bifurcation analysis of families of singly symmetric periodic solutions was provided earlier (see [2, 1]. Here we give such analysis for doubly symmetric solutions.

In the case $S = 1$ there are two possibilities. Either matrix M has only one elementary divisor $(\lambda - 1)^4$, or it has two elementary divisors $(\lambda - 1)^2$ and $(\lambda - 1)^2$. The first one corresponds to a fold (saddle-node) bifurcation of periodic solution at which the family riches the extrema of H. The second one corresponds to a pitch-fork bifurcation, where a pair of singly symmetric periodic solutions appears.

In the case $S = -1$ period doubling bifurcation takes place. For singly symmetric solution the matrix M has 2 elementary divisors $(\lambda - 1)^2$ and $(\lambda + 1)^2$. The eigenvector corresponding to the first elementary divisor gives the direction of continuation of the initial family of periodic solutions. The second eigenvector gives the direction of continuation of a new family of periodic solutions with period $T' = 2T$. This family has the extremum on value $H(Z)$ and preserves the type of symmetry of the initial family.

Let consider the case of doubly symmetric periodic solution.

Statement 1. *The monodromy matrix M of doubly periodic solution in the case $S = -1$ always has 3 elementary divisors $(\lambda - 1)^2$, $\lambda + 1$ and $\lambda + 1$. Eigenvector, corresponding to the first elementary divisor gives the continuation of initial family. Eigenvectors, corresponding to the divisors $\lambda + 1$, give the continuation of the family of double periodic solutions but singly symmetric each. These double periodic solutions have different types of symmetry. The families of double periodic solutions*

rich the extremum of $H(\mathbf{z})$ at the bifurcation point. There are two scenarios of period doubling bifurcation in the case of doubly periodic solutions.

- *Each of two new families has the same type of extremum at the bifurcation point (minimum or maximum). In this case both new families of singly symmetric solutions with period $T' = 2T$ exist near the initial family of doubly symmetric solutions.*
- *Two new families has different types of extremum and thus at each value of the family parameter h there exist the initial family of doubly symmetric solutions and only one new family of singly symmetric solutions with certain type of symmetry.*

Both these scenarios were investigated for periodic solutions of the Hill problem [3] and for periodic solutions of the restricted three body problem in the case of equal masses. Some new families of periodic solutions were found and were studied as well.

The previous situation is a special case of period multiplying bifurcation.

Statement 2. *Let doubly symmetric periodic solution with period T has the stability index $S = \cos 2\pi p/q$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$.*

- *If both p and q are odd, then there exists in vicinity of initial solution one family of doubly symmetric periodic solutions with period $T' = qT$;*
- *If at least one of the numbers is even, then there exist four families mutually pairwise symmetric singly symmetric periodic solutions with period $T' = qT$.*

In all cases except the case $p/q = 1/3$, new families reach the extremum on $H(\mathbf{z})$ at the branching point.

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On computer algebra aided numerical solution of ODE by finite difference method

Yury A. Blinkov and Vladimir P. Gerdt

Except very special cases, nonlinear ordinary differential equations (ODE) admit numerical integration only. Historically first and one of the most-used numerical methods is finite difference method (FDM) [1] based on a finite difference approximation (FDA). As this takes place, the quality of numerical solution to PDE is determined by the quality of its FDA and by the method of numerical solution used to solve the difference equations comprising FDA.

One of the most challenging problems in to construct FDA which mimics basic algebraic properties of the ODE. Such mimetic FDA are more likely to produce highly accurate and stable numerical results (cf. [2]). In particular, a mimetic FDA is to be *totally conservative* (see [3], Def.1) what means the inheritance of algebraic integrals of the ODE at the discrete level.

Example. We consider the following autonomous ODE system [3]

$$\begin{cases} \dot{p} = qr, \\ \dot{q} = -pr, \qquad k = \text{const}\,, \\ \dot{r} = -k^2 pq \end{cases} \tag{1}$$

which has two quadratic integrals

$$p^2 + q^2 = \text{const} \quad \text{and} \quad k^2 p^2 + r^2 = \text{const}. \tag{2}$$

We use the denotations $\mathbf{x} := \{p, q, r\}$ and $\mathbf{F}(\mathbf{x}) := \{qr, -pr, -k^2 pq\}$ and consider the implicit midpoint finite difference discretization of system (1)

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \Longrightarrow \quad \frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{\Delta t} = \frac{\mathbf{F}(\mathbf{x}_{n+1}) + \mathbf{F}(\mathbf{x}_n)}{2} \tag{3}$$

It is known [4] that the scheme (3) preserves integrals (2). Instead of application of the Gröbner bases technique for solution of algebraic system (3) for transition to the next layer, as done in [3], we suggest computationally much more efficient the *simple iteration method*.

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The construction of a simple iteration is based on the following quadratic formulas for monomials occurring in $\mathbf{F}(\mathbf{x})$

$$\begin{aligned} u^2 &= u^2 - u'^2 + u'^2 = (u-u')(u+u') + \\ &\qquad + u'^2 \approx (u-u')2u' + u'^2 = 2uu' - u'^2 \\ uv &= (u+v)^2 - (u-v)^2)/4 \approx ((2(u+v)(u'+v') - (u'+v')^2) - \\ &\quad - (2(u-v)(u'-v') - (u'-v')^2))/4 = uv' + u'v - u'v' \\ u^3 &= u^3 - u'^3 + u'^3 = (u-u')(u^2+uu'+u'^2) + \\ &\qquad + u'^3 \approx (u-u')3u'^2 + u'^2 = 3uu'^2 - 2u'^3 \end{aligned} \tag{4}$$

and the qubic one

$$\begin{aligned} u^2v = uuv &\approx u'u'v + uu'v' + u'uv' - \\ &- (3-1)u'^2v' = u'^2v + 2uu'v' - 2u'^2v' \end{aligned} \tag{5}$$

The rule (5) can be easily implemented in a user's programming language of any modern computer algebra system and allows obtain the code for numerical computations. Thus, for the system (1) and its implicit scheme (3) the simple ireration method yields

$$\begin{cases} \dfrac{p_{n+1}-p_n}{\Delta t} - \dfrac{q_{n+1}r'_{n+1} + q'_{n+1}r_{n+1} - q'_{n+1}r'_{n+1} + q_nr_n}{2} = 0, \\ \dfrac{q_{n+1}-q_n}{\Delta t} - \dfrac{-p_{n+1}r'_{n+1} - p'_{n+1}r_{n+1} + p'_{n+1}r'_{n+1} - p_nr_n}{2} = 0, \\ \dfrac{r_{n+1}-r_n}{\Delta t} - \dfrac{-k^2p_{n+1}q'_{n+1} - k^2p'_{n+1}q_{n+1} + k^2p'_{n+1}q'_{n+1} - k^2p_nq_n}{2} = 0 \end{cases} \tag{6}$$

In the matrix form obtained by using the computer algebra system *SymPy* (`https://www.sympy.org`) is given by

$$\begin{aligned} &\begin{pmatrix} 1/\Delta t & -r'_{n+1}/2 & -q'_{n+1}/2 \\ r'_{n+1}/2 & 1/\Delta t & p'_{n+1}/2 \\ k^2q'_{n+1}/2 & k^2p'_{n+1}/2 & 1/\Delta t \end{pmatrix} \begin{pmatrix} p_{n+1} \\ q_{n+1} \\ r_{n+1} \end{pmatrix} = \\ &\qquad = \begin{pmatrix} p_n/\Delta t - q'_{n+1}r'_{n+1}/2 + q_nr_n/2 \\ q_n/\Delta t + p'_{n+1}r'_{n+1}/2 - p_nr_n/2 \\ r_n/\Delta t + k^2p'_{n+1}q'_{n+1}/2 - k^2p_nq_n/2 \end{pmatrix} \end{aligned} \tag{7}$$

The numerical results obtained with *SciPy* (`https://www.scipy.org/`) and illustrated by Fig. 1 show that there is no accumulation of numerical error, as distinguished from the output of standard ODE solvers `lsoda`, `vode`, `dopri5`, `dop853` of *SciPy*.

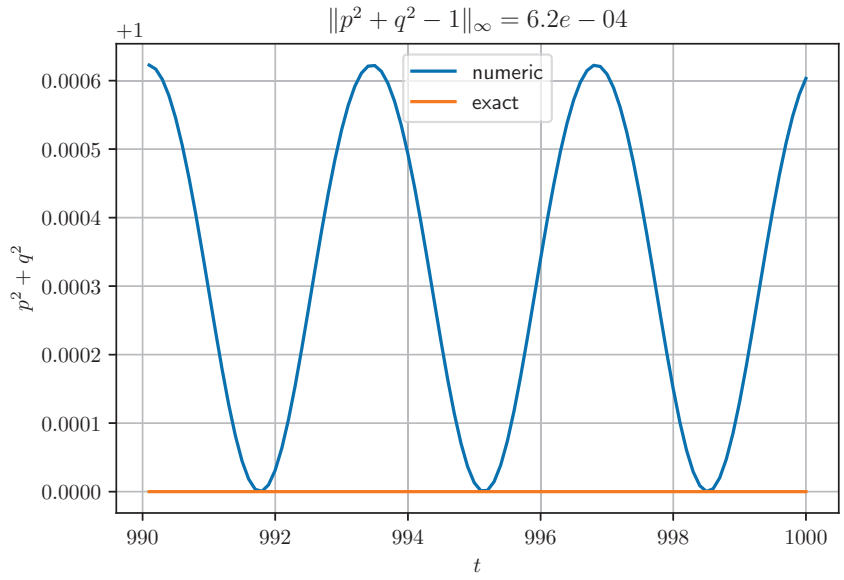

FIGURE 1. Dynamics of numerical error for $p = 0$, $q = r = 1$ and $k = 1/2$

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Applications of Bar Code to involutive divisions and a greedy algorithm for complete sets

Michela Ceria

Abstract. In this paper, given a finite set U of terms in n variables, we see how to compute its Janet decomposition and detect its potential completeness by means of its Bar Code. Moreover, we study an algorithm aimed to return a variables' ordering such that U is complete (or failure if such an ordering does not exist). In principle, one should check all $n!$ orderings, but our algorithm exploits a sort of backtracking technique to skip many of these tests.

Let $\mathbf{k}$ be a field and $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$ the polynomial ring in n variables and coefficients in $\mathbf{k}$. The semigroup of terms generated by $X := \{x_1, ..., x_n\}$ is: $\mathcal{T} := \{x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \,|\, \gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{N}^n\} \subset \mathcal{P}$. In [6], taken a monomial/semigroup ideal $J \subset \mathcal{T}$ and its minimal set of generators $\mathsf{G}(J)$ Janet introduced the notion of *multiplicative variables* and the connected decomposition of J into disjoint *cones*, giving a procedure to construct such a decomposition, by computing a *complete* generating set $H \supseteq \mathsf{G}(J)$ for J, i.e. a particular generating set allowing such a decomposition. In particular, $\forall v \in \mathcal{T}$, there is a *unique* decomposition $v = tu$, with $t \in \mathsf{G}(J)$ and u a product of powers of t's multiplicative variables. While reducing a term w w.r.t. an ideal whose initial ideal is J, we can only use the polynomial whose leading term generates the cone containing w.
Bar Codes [1, 2], are combinatorial objects representing finite sets of terms $M \subset \mathcal{T}$. In particular, if $M = \mathsf{N}(I)$ is the Groebner escalier of a zerodimensional ideal $I \triangleleft \mathcal{P}$, many of its properties can be directly deduced by its Bar Code [1, 2, 3, 4, 5]. In particular, in this paper, we see that Bar Codes are good tools to study Janet decomposition. Given a finite set of terms M, its Bar Code allows us to find the multiplicative variables of its element and to detect its completeness according to Janet's definition. Moreover, we give an algorithm that, using backtracking techniques, allows to find out if there is a variables' ordering such that M is complete without trying all the $n!$ variables' orderings.

1. Janet decomposition via Bar Codes and a greedy algorithm

Referring to [2, 1], we recall first the main facts and definitions about Bar Codes.

Definition 1 ([2, 1]). *A Bar Code* B *is a picture composed by segments, called* bars, *superimposed in horizontal rows, which satisfies conditions a.,b. below. Denote by* $\mathsf{B}_j^{(i)}$ *the* j*-th bar (from left to right) of the* i*-th row (from top to bottom),* $1 \leq i \leq n$*, i.e. the* j-th i-bar, $\mu(i)$ *the number of bars of the* i*-th row,* $l_1(\mathsf{B}_j^{(1)}) := 1$, $\forall j \in \{1, 2, ..., \mu(1)\}$ *the* (1−)length *of the* 1*-bars,* $l_i(\mathsf{B}_j^{(k)})$, $2 \leq k \leq n$, $1 \leq i \leq k-1$, $1 \leq j \leq \mu(k)$ *the* i-length *of* $\mathsf{B}_j^{(k)}$ *(number of* i*-bars lying over* $\mathsf{B}_j^{(k)}$*):*

a. $\forall i, j, 1 \leq i \leq n-1, 1 \leq j \leq \mu(i)$, $\exists! \bar{j} \in \{1, ..., \mu(i+1)\}$ *s.t.* $\mathsf{B}_{\bar{j}}^{(i+1)}$ *lies under* $\mathsf{B}_j^{(i)}$
b. $\forall i_1, i_2 \in \{1, ..., n\}$, $\sum_{j_1=1}^{\mu(i_1)} l_1(\mathsf{B}_{j_1}^{(i_1)}) = \sum_{j_2=1}^{\mu(i_2)} l_1(\mathsf{B}_{j_2}^{(i_2)})$.

We outline how to construct the Bar Code associated to a finite set of terms. First, given a term $t = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \in \mathcal{T} \subset \mathcal{P}$, for each $i \in \{1, ..., n\}$, define $\pi^i(t) := x_i^{\gamma_i} \cdots x_n^{\gamma_n} \in \mathcal{T}$. Taken a set $M \subset \mathcal{T}$, $|M| = m < \infty$, for each $i \in \{1, ..., n\}$, we define $M^{[i]} := \pi^i(M) := \{\pi^i(t) | t \in M\}$. Now we order the elements of M increasingly w.r.t. Lex, getting the list $\overline{M} = [t_1, ..., t_m]$, we construct the sets $M^{[i]}$, and the corresponding lexicographically ordered lists $\overline{M}^{[i]}$, for $i = 1, ..., n$. We can define the $n \times m$ matrix of terms $\mathcal{M}$ s.t. its i-th row is $\overline{M}^{[i]}$, $i = 1, ..., n$. We take then the i-th row of $\mathcal{M}$, consider all the sublists of repeated terms and underline each of them with a segment, deleting the terms of $\overline{M}^{[i]}$, $2 \leq i \leq n$ and leaving only the segments (i.e. the *i-bars*), getting the desired Bar Code.
We recall now Janet's definitions of multiplicative variable and complete set, noting that they depend on the variables' ordering, which in our case is $x_1 < x_2 < ... < x_n$.

Definition 2. [6, ppg.75-9] *Let* $U \subset \mathcal{T}$ *be a set of terms and* $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in U$. *A variable* x_j *is called* multiplicative *for* t *w.r.t.* U *if there is no term in* U *of the form* $t' = x_1^{\beta_1} \cdots x_j^{\beta_j} x_{j+1}^{\alpha_{j+1}} \cdots x_n^{\alpha_n}$ *with* $\beta_j > \alpha_j$. *We denote by* $M_J(t, U)$ *the set of multiplicative variables for* t *w.r.t.* U *and by* $NM_J(t, U)$ *the set of the variables that are not multiplicative for* t *w.r.t.* U non-multiplicative *variables). The* cone *of* t *w.r.t.* U *is the set* $C_J(t, U) := \{t x_1^{\lambda_1} \cdots x_n^{\lambda_n} \,|\, \text{where } \lambda_j \neq 0 \text{ only if } x_j \text{ is multiplicative for } t \text{ w.r.t. } U\}$. *A set of terms* $U \subset \mathcal{T}$ *is called* complete *if for every* $t \in U$ *and* $x_j \in NM_J(t, U)$, *there exists* $t' \in U$ *such that* $x_j t \in C_J(t', U)$; t' *is called* involutive divisor *of* $x_j t$ *w.r.t. Janet division.*

Let $U \subset \mathcal{T} \subset \mathbf{k}[x_1, ..., x_n]$ be a finite set of terms, B its Bar Code. To detect multiplicative variables for the terms in U, $\forall 1 \leq i \leq n$, place a star symbol $*$ on the right of $\mathsf{B}_{\mu(i)}^{(i)}$; then $\forall 1 \leq i \leq n-1$, $\forall 1 \leq j \leq \mu(i) - 1$ let $\mathsf{B}_j^{(i)}$ and $\mathsf{B}_{j+1}^{(i)}$ be two consecutive bars not lying over the same $(i+1)$-bar: place a star symbol $*$ between them. We have

Proposition 3. *Let $U \subseteq \mathcal{T}$ be a finite set of terms and B its Bar Code. For each $t \in U$ x_i, $1 \leq i \leq n$ is multiplicative for t if and only if, in B, the i-bar $\mathsf{B}_j^{(i)}$, over which t lies, is followed by a star.*

Proposition 4. *Let $U \subseteq \mathcal{T}$ be a finite set of terms and B be its Bar Code. Let $t \in U$, $x_i \in NM_J(t, U)$ and $\mathsf{B}_j^{(i)}$ the i-bar under t. Let $s \in U$; $s \mid_J x_i t$ iff*

1. $s \mid x_i t$
2. s lies over $\mathsf{B}_{j+1}^{(i)}$
3. $\forall j'$ with nonzero exponent in $\frac{x_i t}{s}$ there is a star after the j'-bar under s.

Theorem 5. *Let $U \subseteq \mathcal{T}$ be a finite set of terms and B be its Bar Code. Then U is a complete set if and only if $\forall t \in U$, $\forall x_i \in NM_J(t, U)$, called $\mathsf{B}_j^{(i)}$ the i-bar under t, there exists a term $s \in U$ which satisfies conditions $1, 2, 3$ of Proposition 4.*

According to Proposition 4 and Theorem 5, given a finite set of terms $U \subseteq \mathcal{T}$, to check its completeness we take, $\forall t \in U$, $\forall x_i \in NM_J(t, U)$, the i-bar $\mathsf{B}_j^{(i)}$, $1 \leq j \leq \mu(i)$ under t and we look for an involutive divisor among the terms over $\mathsf{B}_{j+1}^{(i)}$, checking conditions 1,2,3 above. Such a construction depends on the variables' ordering, so if we want to find out whether there is one making a given set complete, in principle, we should draw and check $n!$ different Bar Codes, a tedious and time consuming task. We can look for the solution of our problem in a "greedy" way, so that most of the tests can be skipped, by means of the Bar Code and of

Corollary 6 ([6, 7])**.** *Let $U = \{t_1, ..., t_m\} \subseteq \mathcal{T}$ be a finite set of terms, $t_i = x_1^{\alpha_1^{(i)}} \cdots x_n^{\alpha_n^{(i)}}$ and $t_i' = x_1^{\alpha_1^{(i)}} \cdots x_{n-1}^{\alpha_{n-1}^{(i)}} = \frac{t_i}{x_n^{\alpha_n^{(i)}}}$, for $i = 1, ..., m$. Let $U' = \{t_1', ..., t_m'\}$, $\alpha = \max\{\alpha_n^{(i)},\ 1 \leq i \leq m\}$. For each $\lambda \leq \alpha$, $I_\lambda := \{i : 1 \leq i \leq m | \alpha_n^{(i)} = \lambda\}$ is the set indexing the terms in U with n-th degree equal to λ, and $U'_\lambda := \{t_i' | i \in I_\lambda\}$. Then U is complete if and only if the two conditions below hold:*

1. *For each $\lambda \in \{\alpha_n^{(i)},\ 1 \leq i \leq m\}$, U'_λ is a complete set;*
2. *$\forall t_i' \in U'_\lambda$, $\lambda < \alpha$, there exists $j \in \{1, ..., m\}$ s.t. $t_i' \in C_J(t_j', U')$ and $t_j' \in U'_{\lambda+1}$.*

The idea consists in constructing the Bar Code B of the set $U \subset \mathcal{T}$ from the maximal variable to the minimal one, checking if, with the ordering choices made up to the current point, the conditions of Proposition 4 hold for each term in U, and going back retracting our steps in case of failure. In the first step we look for the subset $Y \subseteq X$ of good candidates for being the maximal variable. For each variable x_i, $1 \leq i \leq n$, we compute the set D_i of degrees $\beta \in \mathbb{N}$ s.t. x_i^β appears as exponent of some term in U and we have $x_i \in Y$ if and only if D_i contains only consecutive elements (Corollary 6). If $Y = \emptyset$, no variable can be the maximal one, making U complete, so U is not complete for any variables' ordering. Suppose instead $\emptyset \neq Y = \{x_{j_1}, ..., x_{j_l}\} \subseteq X$; we pick $x_{j_1} \in Y$ as maximal variable. We reorder the elements of U, increasingly w.r.t. their j_1-degree and if $t, t' \in U$ have the same j_1-degree and $t \mid t'$, we set $t < t'$; then we construct the j_1-bars $\mathsf{B}_1^{(j_1)}, ..., \mathsf{B}_{\mu(j_1)}^{(j_1)}$ under the terms. Now we look for candidate terms for having condition 2 of Corollary 6 satisfied, seeking in the Bar Code the potential

involutive divisors of all tx_{j_1}, for t over $\mathsf{B}_1^{(j_1)}, ..., \mathsf{B}_{\mu(j_1)-1}^{(j_1)}$ and keeping track of the variables they would need to have as multiplicative to really be the involutive divisors. If some of these terms has no candidate involutive divisors, then x_{j_1} is not a good candidate for being the maximal variable, so we come back to Y and we start again with a new one. Otherwise, if for $1 \leq j \leq \mu(j_1)$ there is only one term over $\mathsf{B}_j^{(j_1)}$, all the bars are *unitary* (*unitary case*): each variables' ordering s.t. x_{j_1} is the maximal one makes U complete. If we are not in the unitary case, we have to choose the next variable and continue drawing the Bar Code. To get the candidates for being the next variable, we look for candidates from each j_1-bar and we intersect the results, coming back and repeating the whole procedure with a new maximal variable if that intersection is empty. We choose then some x_{j_2} among the variables in the intersection, and for each $1 \leq j \leq \mu(j_1)$, we order the terms over $\mathsf{B}_j^{(j_1)}$ exactly as done for constructing the j_1-bars and we draw all the j_2-bars. We look for candidate involutive divisors when x_{j_2} is not multiplicative, separately for each j_1-bar. Moreover, we check whether the choice of x_{j_2} is suitable to the candidates found in the previous step, removing the candidates such that x_{j_2} is not suitable for them. If for some t its candidate list is empty we have to revoke the choice of x_{j_2} and come back with another candidate. If the procedure gives a positive outcome, then a new variable has been chosen and we continue this way until either all variables have been placed (positive outcome), or the unitary case is reached (positive outcome), or continue revocations of choices lead to failure (negative outcome).

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Visual Data Explorer

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Dynamical systems modeling often produces large sets of (multivariate) data, so modern computer algebra systems have a good visualization tools. Data visualization plays an important role in data analysis - visual inspection can help to reveal patterns that would be computationally rather difficult to reveal. The problem becomes especially involved when one deals with multivariate data. Unfortunately, not all computer algebra systems have good interactivity tools, and analyzing structures algorithmically is not easy. When we encountered this problem, it was decided to make a system for interactive exploration of multivariate data.

The system is designed to allow the user to find relations between two projections of a large multivariate dataset. Two selected variables are displayed on a scatter plot (left plot) with another two selected variables displayed on another scatter plot (right). The system can be scaled horizontally to produce as many plots as required from the same initial dataset. Points can then be selected on any plot and corresponding points will be selected automatically on the other plots. Figure 1 gives an example of such visualization and selection. Selected cases can be saved into a file for further analysis.

This system was initially developed using Microsoft Visual Studio IDE with Python and some of its supporting libraries designed for data analysis and visualization. Later, the system was migrated to Jupyter Notebooks in order to fully utilize the power of PyViz libraries specifically developed for this application. The main libraries imported were: Pandas; an open-source Python library designed for data science applications, and Holoviews with Bokeh on the backend; a Python library designed specifically for interactive data visualization using web browsers through JavaScript. Also Tkinter; to create a graphical user interface and accept user input, and NumPy; to effectively and efficiently perform mathematical operations if needed.

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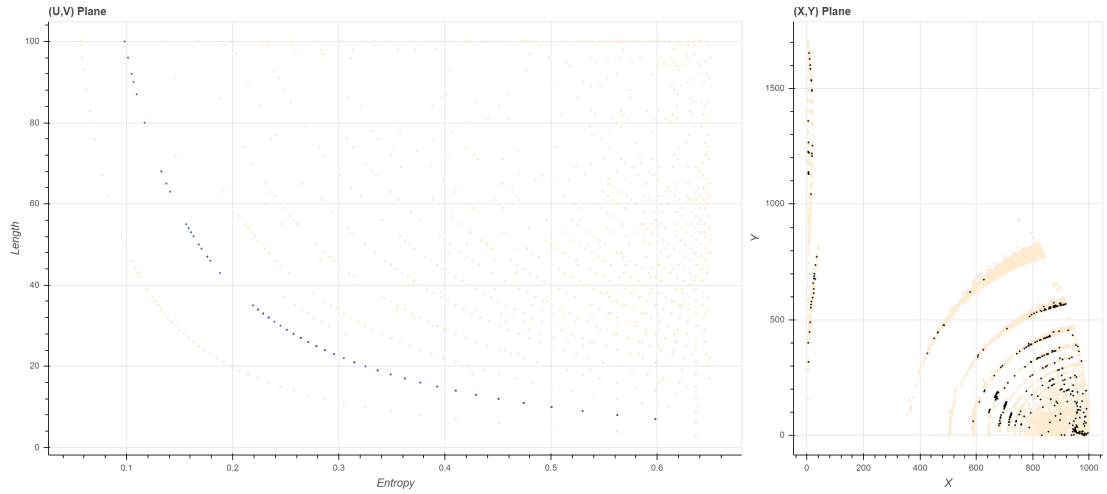

FIGURE 1. Left: One curve observed in the (U, V) projection is selected. Right: corresponding projection in another set of coordinates.

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Developing a Wolfram Demonstrations Project for bifurcation diagrams of nonlinear ODE

Daria Chemkaeva and Alexandr Flegontov

Abstract. The study considers creating a project using computing system Wolfram Mathematica 11.3 for some classes of nonlinear boundary value problems (BVP) including a second-order autonomous ordinary differential equation with homogeneous boundary conditions. Here is studied a general case applying to polynomial-like nonlinearities. Developing a Wolfram Demonstrations Project (WDP) for this problem will accelerate experiments and decrease computational time because of no need of sequential execution of algorithm.

Introduction

We study the interconnection of bifurcation parameter and positive roots of the nonlinear two-point boundary value problem:

$$y''_{xx} + \lambda f(y(x)) = 0, \quad x \in (-1; 1), \tag{1}$$

$$y(-1) = y(1) = 0. \tag{2}$$

Assume $f = f(y)$ so second order ODE is autonomous, where parameter $\lambda > 0$. In this case, the bifurcation arises when the number of solutions of the differential equation changes as the parameter λ changes. Assume also that $f = f(y)$ is a polynomial of odd degree. The problem (1)–(2) describes many physical processes, for example, belongs to the problems of combustion of gases, population dynamics and reaction-diffusion models.

Section 1 is technical and contains useful supplement of BVP and bifurcation curve. Section 2 describes the capabilities of system Wolfram Mathematica 11.3 in creating demonstration projects. Section 3 is the main part of the study where the programming algorithm is described.

1. Nonlinearity as a polynomial of odd degree

In previous study [1] we found that bifurcation curve can be written in the form:

$$\lambda(a) = \frac{1}{2}\left[\int_0^a \frac{dt}{\sqrt{F(a) - F(t)}}\right]^2, \tag{3}$$

where $F(y) = \int_0^y f(t)\,dt$ and $F(a) \geq F(t)$.

The turning points of (3) are bifurcation points. The plot of this function is called bifurcation diagram [2], implying an image of change in the possible dynamic modes of the system with a change in the value of bifurcation parameter $\lambda > 0$.
We also assumed that function $f(y)$ is a polynomial of odd degree, and consequently can change the sign:

$$f(y) = (y - a_1)(y - a_2)(y - a_3)...(y - a_{2n-2})(a_{2n-1} - y), \tag{4}$$

where $0 < a_1 < a_2 < ... < a_{2n-2} < a_{2n-1}$ – isolated zeros of function $f(y)$, i. e. $f(a_i) = 0$, so the problem (1)–(2) has trivial solutions:

$$y = a_i, \quad i = 1,\, 2,\, \ldots,\, 2n - 1. \tag{5}$$

Function (4) has odd number of zeros and is negative on (a_1, a_2), then it is positive on (a_2, a_3). Therefore, the function has n pairs of humps, where $f(y) > 0$ on $(a_{2n-2}), a_{2n-1})$ and $f(y) < 0$ on (a_{2n-3}, a_{2n-2}).
BVP (1)-(2) with $f(y)$ in form (4) has λ_i $(i = 0...n)$ bifurcation parameters such that for $\lambda < \lambda_0$ there is one solution to the problem, for $\lambda = \lambda_0$ – exactly two solutions, for $\lambda_0 < \lambda < \lambda_1$ – three solutions and so on. Such parameters will only be found in the intervals, where $f(y) > 0$. The maximum number of positive solutions always depends on the degree of the polynomial and equals n [3].
Generalization of this problem leaded us to development of special control in Wolfram Mathematica language to have an interactive tool for studying the interconnection between the number of roots of $f(y)$, number of positive solutions and bifurcation parameters.

2. Wolfram Demonstrations Project

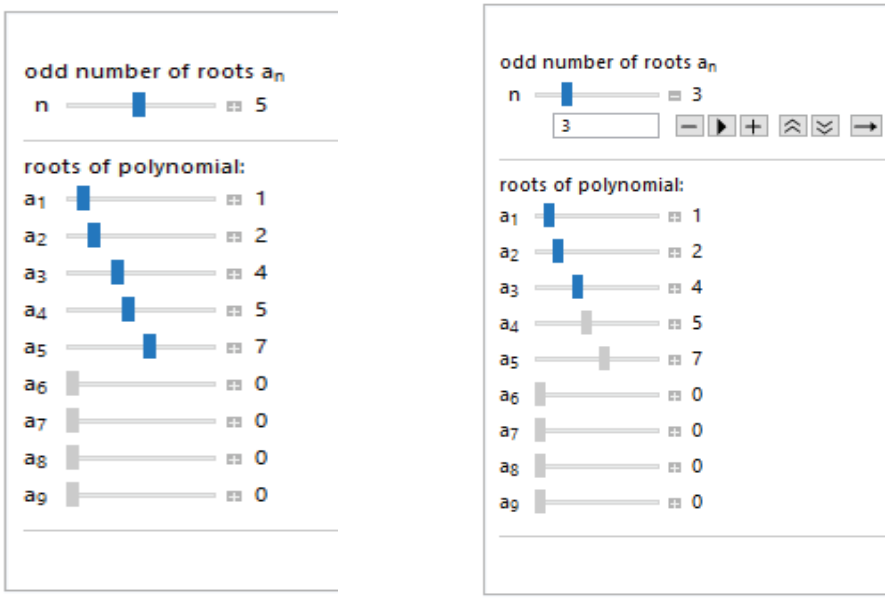
The Wolfram Demonstrations Project (WDP) is an open-source collection of compact interactive programs called Demonstrations, which are meant to visually and interactively represent ideas from a range of fields. It is hosted by Wolfram Research, whose stated goal is to bring computational exploration to the widest possible audience.
All demonstrations run freely on any standard Windows, Mac or Linux computer and there is no need to have Mathematica pre-installed. To interact with any Demonstration user can work with free Wolfram CDF Player right in web browser. If the Mathematica is installed, any source code can be downloaded, operated and

modified [4]. The Wolfram Demonstrations Project community has a huge variety of interactive visualizations including mathematics, physics, computer science, business and even creative arts topics.
WDP consists of a title, Manipulate code (including initialization) which represents an interactive part of a demonstration, caption and thumbnail which describe a project.

3. Developing a Manipulate code

The programming module was developed using a single command Manipulate which output is an interactive object containing one or more controls that can be used to vary the value of parameter list. The output of Manipulate is very much like a small widget: it is not just a static result, it is a running program which can be interacted with.
The interactive part of Manipulate command is a set of control elements (sliders) for:

1. Odd number of roots n ($n = 5$ by default);
2. Roots of polynomial (4) a_i. Number of active sliders for roots depends on n. Changing the value of n leads to activation / deactivation of bottom sliders (Fig. 1).



(a) $n = 5$ (b) $n = 3$

FIGURE 1. Slider controls for input parameters

The evaluation part of Manipulate control consists of following steps (Fig. 2-3):

1. Initialize function (3) of numeric parameter a;
2. Input number of roots of polynomial (4) and exact values of roots which are positive numbers with the step 0.1;
3. Sort roots in ascending order and find a root with maximal value (it helps to organize plot axis and grid lines);
4. Create a polynomial function (4);
5. Create an integral function $F(y) = \int_0^y f(t)\,dt$;
6. Find the turning points of (3) using $NMinimize$;
7. Get the array of turning points a and array of $\lambda(a)$ using $Table$;
8. Plot bifurcation curve (3) in coordinate system (a, λ) with an array of $\lambda(a)$ as lines using $Plot$ command;
9. In $Epilog$ parameter of $Plot$ register roots of polynomial a_i (black points), turning points a (red points) and bifurcation parameters λ (ticks on λ-axis).

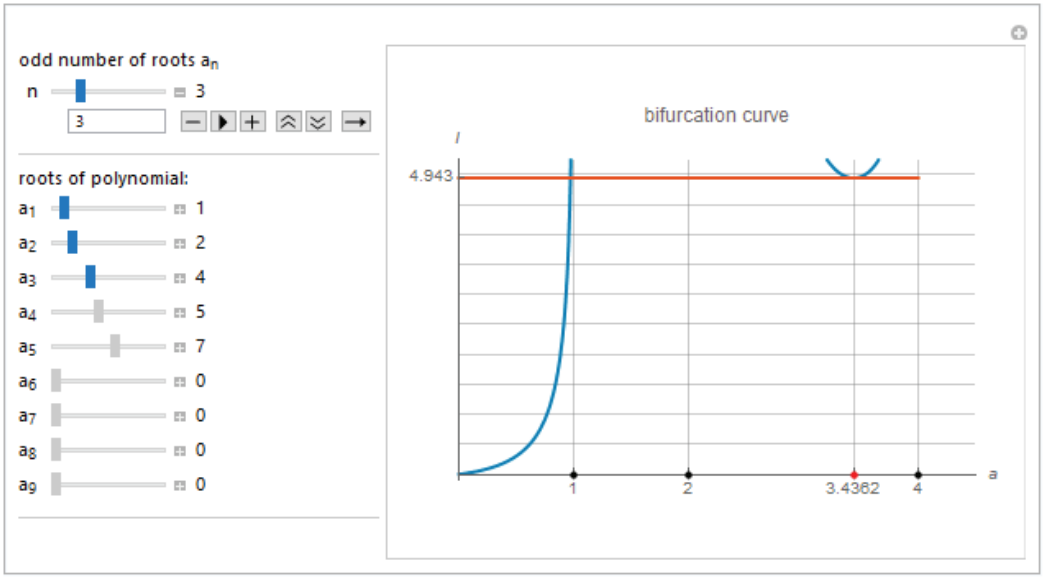


FIGURE 2. Manipulation Control (sample 1)

Conclusion

With the help of interactive Wolfram Demonstrations Project is obvious that problem (1)–(2), where $f(y)$ – polynomial of odd degree in form (4) has exactly λ_i $(i = 0...n)$ bifurcation parameters and the maximum number of positive solutions always depends on the degree of the polynomial and equals λ_n. With the help of Wolfram interactive control we have the tool for further analysis of behavior and interconnection between the number of roots of $f(y)$, number of positive solutions of BVP and bifurcation parameters.

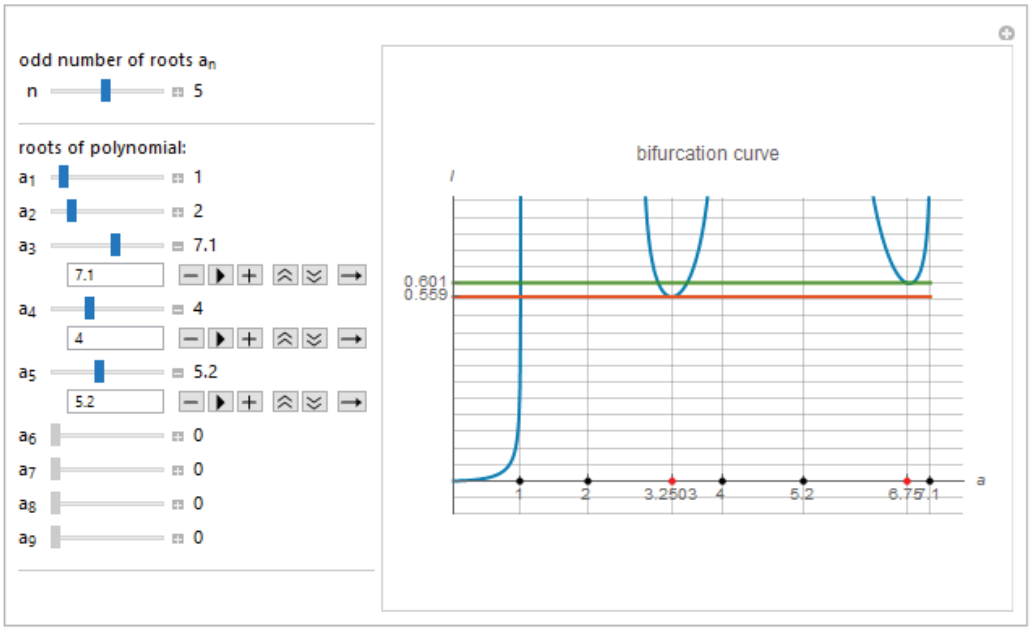


FIGURE 3. Manipulation Control (sample 2)

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A Numerical Roadmap Algorithm for Smooth Bounded Real Algebraic Surface

Changbo Chen, Wenyuan Wu, Yong Feng

Abstract. For a smooth bounded real algebraic surface in three-dimensional space, a roadmap of it is a one-dimensional semi-algebraic subset of the surface whose intersection with each connected component of the surface is nonempty and semi-algebraically connected. In this paper, we introduce the notion of a numerical roadmap of a surface, which is a set of polygonal chains such that there is a bijective map between the chains and the connected components of a given roadmap of the surface. Moreover, the chains are ϵ-close to the connected components. We present an algorithm to compute such a numerical roadmap through constructing a topological graph. The topological graph also enables us to compute a more intrinsic connectivity graph of the roadmap, which is important for applications such as finding a connected path between two points on the surface, as well as grouping witness points of the surface into different connected components.

Introduction

Roadmap was introduced by Canny [6] in 1987 for solving robot motion planning problems. Since then, the initial roadmap algorithm has been improved by himself and many others [17, 14, 11, 20]. For a polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree d, Basu et al. [3] recently proposed a symbolic roadmap algorithm with a complexity of $d^{O(n\sqrt{n})}$ based on the earlier work of Safey el Din and Schost [10].

The problem of deciding whether two points belong to the same connected component of a semi-algebraic set is a fundamental problem in real algebraic geometry. Such problems can be solved by computing cylindrical algebraic decompositions (CAD) [9, 21], Today, the implementations of CAD are widely available thanks to different softwares QEPCAD, Mathematica, REDLOG, SyNRAC, `RegularChains`. The complexity for computing CAD is double exponential in the number of variables. In contrast, the roadmap based algorithms have a single exponential complexity and thus provides a theoretically more powerful tool for solving

the connectivity problems in semi-algebraic geometry, such as determining if two points in a semi-algebraic set are connected, or counting the connected components of a semi-algebraic set. However, to the best of our knowledge, there have been no exact implementations of the roadmap algorithms.

Today, many problems in computational real algebraic geometry can be addressed in a different way by means of powerful tools from numerical algebraic geometry [22, 13], such as computing witness points for connected components of real varieties [12, 24, 23]. Despite some attempts [15, 16] for numerically computing roadmaps, the notion of numerical roadmap has not been rigorously defined until now. And there are still no complete numerical implementations of the roadmap algorithms.

It is natural to develop a numerical roadmap algorithm since the roadmap as a one-dimensional semi-algebraic set can be approximated by polygonal chains based on numerical continuation techniques. The difficulty for developing such an algorithm would be to guarantee there is a one-to-one correspondence between the connected components of the roadmap and the connected components of its polygonal chains approximation. Indeed, to achieve this, from a numerical point of view, one has to overcome some obstacles, such as avoiding curve jumping [5, 4, 19, 25, 24, 7] and handling singularities [1, 18, 8].

In this work, for a given smooth bounded real algebraic surface, we provide a numerical version of the classical roadmap algorithm introduced by Canny [6, 2]. We introduce the concept of a numerical roadmap and propose an algorithm to generate it through constructing some graphs joining the silhouette of the surface and slice curves passing through the critical points of a projection map on the silhouette. The slice curves may have singularities, which is handled based on a technique for tracing singular planar curves [7].

1. Main results

Throughout this paper, let $f \in \mathbb{R}[X_1, X_2, X_3]$ and $\mathrm{Z}_{\mathbb{R}}(f)$ (or simply Z if no confusion arises) be its zero set in $\mathbb{R}^3$. We assume:

(A_1) $\mathrm{Z}_{\mathbb{R}}(f)$ is nonempty[1] and bounded.

(A_2) f attains full rank at any point of $\mathrm{Z}_{\mathbb{R}}(f)$.

Thus, $\mathrm{Z}_{\mathbb{R}}(f)$ is a smooth bounded surface in $\mathbb{R}^3$. In addition, without lost of generality, we enforce the assumption (A_3): the critical set of π_{12} is a manifold, holds.

Definition 1. *A one-dimensional semi-algebraic subset RM of* Z *is called a* **roadmap** *of* Z *if the following two properties are satisfied:*

(R_1) *The intersection of* Z *with each semi-algebraically connected component of RM is nonempty and semi-algebraically connected.*

(R_2) *For every* $c \in \mathbb{R}$*, every semi-algebraically connected component of* Z_c *has nonempty intersection with RM.*

[1] If $\mathrm{Z}_{\mathbb{R}}(f)$ is empty, its roadmap will be empty. We make this assumption for simplicity.

Definition 2. *The critical set* $\Sigma(\pi_{12}|\mathrm{Z})$ *is called the* **silhouette** *of* Z. *For any* $c \in \mathbb{R}$, *if* $\mathrm{Z}_c \neq \emptyset$ *and* $\dim(\mathrm{Z}_c) \leq 1$, *it is called a* **slice curve** *of* Z.

Let SI be the silhouette of Z. By Sard's Theorem, there are only finitely many critical values of $X_1 : SI \to \mathbb{R}$. Let $c_1 < \cdots < c_r$ be all the X_1-critical values of SI, where c_1 and c_r are respectively the minimal and maximal value of X_1 on Z. Let $SL_0 := \cup_{i=2}^{r-1} \mathrm{Z}_{c_i}$ and $RM_0 := SI \cup SL_0$.

We call each hyperplane $X_1 = c_i$, $i = 2, \ldots, r-1$ a **distinguished hyperplane** and each point in $\cup_{i=2}^{r-1} \mathrm{Z}_{c_i} \cap SI$ a **distinguished point**.

Theorem 1 ([6, 2]**).** *The set* RM_0 *is a roadmap of* Z.

Definition 3. *Given a roadmap* RM *of* $\mathrm{Z}_{\mathbb{R}}(f)$ *and a given precision* $\epsilon \in R$. *A set* $\mathcal{S}$ *of polygonal chains is called a* **numerical roadmap** *of* $Z_{\mathbb{R}}(f)$ *(*ϵ*-close to* RM*) if there is a bijection map* $\mathfrak{m}$ *between* $\mathcal{S}$ *and the connected components of* RM *such that the Hausdorff distance* $d_H(P, \mathfrak{m}(P)) \leq \epsilon$ *holds for each* $P \in \mathcal{S}$.

Note that if we make a small ϵ perturbation to a surface Z, the number of connected components of Z may change if the Hausdorff distance between two components is less than ϵ. Thus, in the rest of this paper, we would assume that the Hausdorff distance between any two connected components of a surface is much larger than ϵ. Moreover, we assume that the distance between any X_1-critical points of the silhouette is much larger than ϵ. We name the two numerical assumptions as (A_4).

The main idea is to use the distinguished points as well as the critical points on the hyperplane $X_1 = c_1$ and $X_1 = c_r$, which are the X_1-global extremum points on the surface, as seed points to generate some neighbor points. These neighbor points will be used as initial points for curve tracing. The distinguished points, critical points on the hyperplane $X_1 = c_1$ and $X_1 = c_r$ and neighbor points will be the vertices of the graph and they are connected by edges representing the curve segments between them. We call this graph a **topological graph** of the roadmap RM_0. We can refine this graph to obtain an **approximate graph** of RM_0 and build the **connectivity graph** of RM_0.

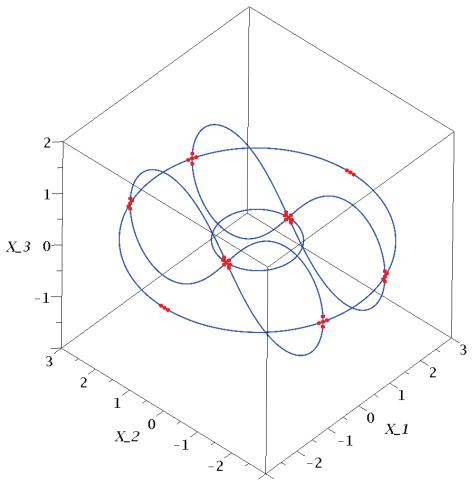
Theorem 2. *Under Assumptions* $(A_1), \ldots, (A_4)$, *one can control errors of staring points and prediction-correction in curve tracing to compute an* **approximate graph** *of* RM_0, *whose zero set is a numerical roadmap* ϵ*-close to the roadmap* RM_0 *of* Z.

2. Examples

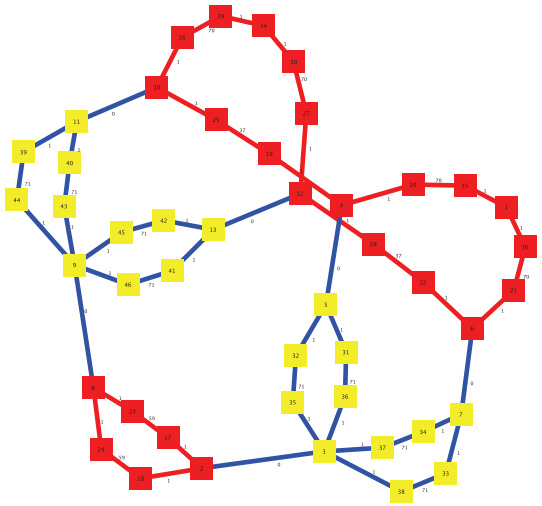
Example 1. *Consider the torus surface defined by*

$$f := \left(x^2 + y^2 + z^2 + 3\right)^2 - 16\,x^2 - 16\,y^2.$$

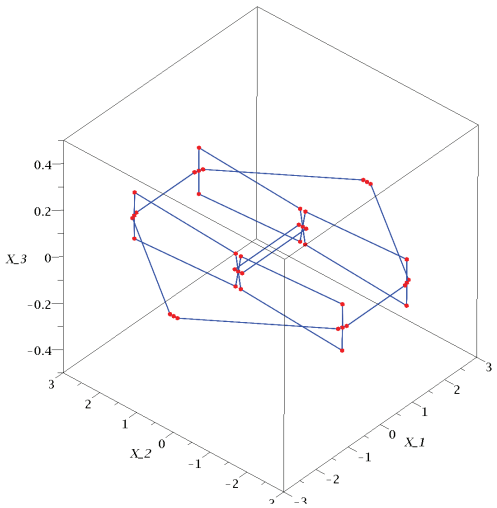
Example 2. *The Chub's surface defined by* $f := x^4 + y^4 + z^4 - x^2 - y^2 - z^2 + 0.5$ *is a singular surface. We replaces the constant coefficient of* f *by* 0.4 *(resp.* 0.6*) and the perturbed polynomials as* $Chub_1$ *(resp.* $Chub_2$*).*



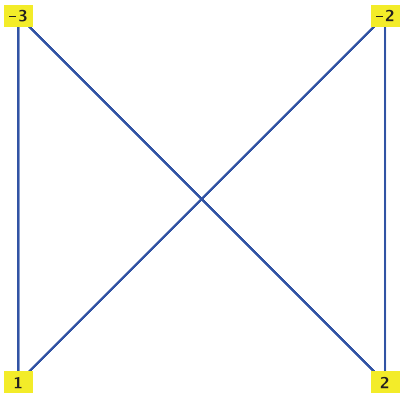
A numerical roadmap of the torus.



The computed topological graph of the torus.



Zero set of the topological graph of the torus.



The connectivity graph of the torus.

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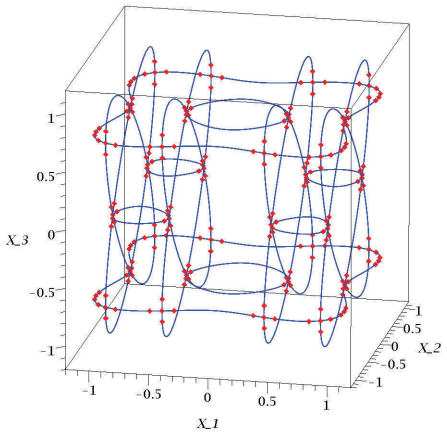
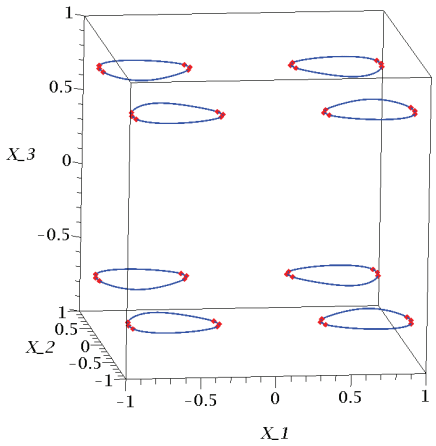


A numerical roadmap of $Chub_1$.



A numerical roadmap of $Chub_2$.

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Problems around the Newton-Puiseux algorithm and its generalization to nonzero characteristic

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Abstract. We discuss problems and recent results related to the Newton-Puiseux algorithm and its generalization for nonzero characteristic obtained by the author earlier. In particular we suggest explicit expansions of algebraic functions in formal power series in nonzero characteristic and thereby obtain an interesting generalization of the Taylor theorem to nonzero characteristic for these functions.

In paper [3] we developed a new method and proved the fundamental result in theory of computation with parameters. After that we demonstrate the strength of this method in [4], [5], [6] (the third concluding part of [4], [5] is to appear) and solved there long standing difficult problems.

It is interesting now to apply the methods of [3], [4] to the problem of constructing Newton-Puiseux expansions of the roots of polynomials with parametric coefficients. Of course here one should work with commutative separable algebras over the ground field in place of its finite field extensions. It is only one of the difficulties. To obtain good bounds for the complexity one can apply the results of [8]. We hope to consider this problem in detail in one of our next papers.

In [7] we generalized the Newton-Puiseux algorithm to the case of a nonzero characteristic ground field k. There we obtained a canonical algorithm for factoring polynomials over the maximal weakly ramified extension of the field $k((X))$. Note that so far there has been a general opinion that such an algorithm is impossible or if it exists it must be very complicated. So our result from [7] can be considered as a true discovery. Still there is a problem to estimate the sizes of coefficient from finite extensions of the field k involved in this natural construction similarly to [8] (there the field k has zero characteristic). But now it seems there is no direct analog of the results from [8] sufficient to get the required bounds for sizes of coefficients. We need to return to a more classical approach and estimates the denominators of these coefficients. More precisely, we would like to formulate the following hypothesis.

Let $k = \mathbb{F}_{p^m}(t_1, \dots, t_l)$ where $t_1, \dots, t_l$ are algebraically independent over the finite field $\mathbb{F}_{p^m}$ of p^m elements, $p = \mathrm{char}(k)$, $m \geqslant 1$ is an integer. Denote by k_s the separable closure of the field k. Let $f \in k[X, Y]$ be a separable polynomial with respect to Y (i.e., the degree $\deg_Y f = n \geqslant 1$ and the discriminant of f with respect to Y is nonzero) with the leading coefficient $\mathrm{lc}_Y f = f_n$. Assume that $f \in \mathbb{F}_{p^m}[t_1, \dots, t_l, X, Y]$ and the degrees $\deg_{X,Y} f \leqslant d$, $\deg_{t_1,\dots,t_l} f \leqslant d_1$ for some $d, d_1 \geqslant 2$. Let $g \in k_s((X))[Y]$ be an irreducible (in the last ring) factor of the polynomial f. Then by this hypothesis there are a polynomial $0 \neq \lambda \in \mathbb{F}_{p^m}[t_1, \dots, t_l]$ of degree $\deg_{t_1,\dots t_l} \lambda = d_1 d^{O(1)}$ and a polynomial $g_1 = g_1(t_1, \dots, t_l, X, Y) \in \mathbb{F}_{p^m}[t_1, \dots, t_l][[X]][Y]$ such that

$$g = g_1(t_1, \dots, t_l, X/\lambda, Y).$$

One can even specify this hypothesis. Namely, let $y_1, \dots y_n \in \overline{k((X))}$ be all the the pairwise distinct roots of the polynomial f (here $\overline{k((X))}$ is an algebraic closure of the field $k((X))$). Put $F = f_n^{n(n-1)} \prod_{1 \leqslant i \neq j \leqslant n} (Z - y_i + y_j)$ where Z is a new variable. So the polynomial $F \in \mathbb{F}_{p^m}[t_1, \dots, t_l, X, Z]$. Let $F = \sum_{i,j} F_{i,j} X^i Z^j$ where all the coefficients $F_{i,j} \in \mathbb{F}_{p^m}[t_1, \dots, t_l]$. Let V be the set of all vertices of the Newton broken line of the polynomial F considered as an element of $k[[X]][Z]$. Put $\lambda_1 = \prod_{(i,j) \in V} F_{i,j}$. Then one can suppose additionally in the formulated hypothesis that λ divides λ_1^N for some integer $N = d^{O(1)}$.

This hypothesis (if it is true) is a key to obtain good bounds for sizes of coefficients from k_s in the construction from [7]. To prove this hypothesis we need to analyze carefully the algorithm from [7] (at present we don't see any other way).

Notice also that one can generalize the expansions introduced in [7] to obtain the canonical algorithm for factoring polynomials in the ring $k_s((X))[Y]$ in nonzero characteristic (k is arbitrary in what follows). Here in some sense one can combine the expansions from [1], [2] and [7]. Of course the algorithm for factoring polynomials in the ring $k_s((X))[Y]$ can be deduced from the algorithm of [7] immediately without introducing these new expansions but it will not be canonical.

Even more, one can describe further generalization of these expansions and obtain a canonical algorithm for factoring polynomials in the ring $k((X))[Y]$ in nonzero characteristic. But here one should work with commutative separable algebras over the ground field in place of its finite field extensions, cf. above. This last generalization will be useful for explicit algorithms in theory of algebraic curves in nonzero characteristic and allows to obtain the results similar the ones from [9], [10] (in [9] a more general situation is considered and some inaccuracies from [10] are corrected). For example, using these results one can compute the genus of a curve. Notice also that in the case $l = 0$, i.e., if the ground field is finite all these results follow directly from [11] (in the English translation of the last paper two pages 1913, 1914 are given in the wrong order; besides this we have found in the original paper and its translation a small non-essential inaccuracy in the statement of lemma 5).

One more interesting problem related to this subject is to get an efficient algorithm for factoring polynomials over the rings of multi-variable formal power series in any characteristic. Say, to factor a polynomial $\psi \in k[[X_1, X_2]][Y]$ in this ring.

At the end of this extended abstract we would like to describe an explicit analog of the Taylor series for algebraic function in nonzero characteristic. Let k be an arbitrary field of characteristic $p > 0$. Put $K = k((X))$. Let $f \in k[[X]][Y]$ be a separable polynomial with respect to Y with the degree $\deg_Y f = n \geqslant 1$. Consider the separable algebra $A = K[Y]/(f)$. Put $y = Y \bmod f \in A$. Let Z, W be new variables. For every $\varphi \in k[[X]]$ the element $\varphi(X+Z) \in k[[X, Z]] \subset K[[Z]]$ is defined in the natural way. So $f(X + Z, W) \in K[[Z]][W]$. Put $B = K[[Z]][W]/(f(X + Z, W))$ and $w = W \bmod f(X + Z, W) \in B$. So not formally we have $w = y(X + Z) = y|_{X:=X+Z}$. Now we would like to find an embedding of $K[[Z]]$-algebras $K[[Z]][w] \to K[y][[Z]]$ such that $w \mapsto \sum_{i\geqslant 0} w_i Z^i$, where all $w_i \in K[y] = A$ and $w_0 = y$, i.e. to find an explicit representation $w = \sum_{i\geqslant 0} w_i Z^i \in K[y][[Z]]$. Such a representation exists by the inverse function theorem for formal power series over K since we have $f'_Y(X, y)$ is invertible in A due to the separability of the polynomial f with respect to Y. The inverse function theorem is valid in nonzero characteristic and it is deduced (similarly to the case of zero characteristic) from the implicit function theorem for formal power series over $K[y]$ in two variables Z, T with a nonzero invertible Jacobian, namely considering the mapping $(Z, T) \mapsto (Z, f(X + Z, T + y))$. So by definition put $D_i y = w_i$ for every $i \geqslant 0$. Now for every $z \in K[y]$ one can define the elements $D_i z \in K[z] \subset K[y]$ in the similar way. Hence D_i are k-linear operators on $K[y]$. We have $D_i(z_1 z_2) = \sum_{0\leqslant m\leqslant i} D_m(z_1) D_{i-m}(z_2)$ and $D_i(D_j(z)) = \binom{i+j}{i} D_{i+j}(z)$ for all integers $i, j \geqslant 0$ and $z_1, z_2, z \in K[y]$. Notice also that if the initial polynomial $f \in k[X, Y]$ then all $D_i y \in k(X)[y]$ (it is also the consequence of the inverse function theorem).

Of course, in the case of zero characteristic we have $D_i y = \frac{1}{i!} \frac{d^i y}{dX^i}$ by the Taylor theorem. In nonzero characteristic if $y \in k_s((X))$ and $y = \sum_{j\geqslant j_0} a_i X^j$, all $a_j \in k_s$ (in this case $\deg_Y f = 1$), then one can prove that $D_i y = \sum_{j\geqslant j_0} a_j \binom{j}{i} X^{j-i}$.

Let us return to the case of an arbitrary polynomial f with $\deg_Y f \geqslant 1$. At present we would like to get general formulas for $D_i(z)$ for $z \in A$ in nonzero characteristic. Strangely enough we could not find them in literature. Let $s \geqslant 0$ be an integer. We have $K[y] = K[y^{p^s}]$ since f is separable with respect to Y. One can represent $f^{p^s} = f_s(X^{p^s}, Y^{p^s})$ where $f_s \in k[[X]][Y]$ is a separable polynomial with respect to Y. Denote by δ_s the differentiation of the ring $k((X^{p^s}))[Y^{p^s}]$ over k such that $\delta_s(X^{p^s}) = 1$. Such a differentiation exists and unique since the polynomial f_s is separable with respect to Y. Speaking not quite formally it coincides with d/dX^{p^s} on the ring $k[[X^{p^s}]]$. Let $z \in k((X))[y]$. We represent $z = \sum_{0\leqslant i<p^s} z_i X^i$ where all $z_i \in k((X^{p^s}))[y^{p^s}]$. By definition put $\delta_s(z) = \sum_{0\leqslant i<p^s} \delta_s(z_i) X^i \in A$. Here we would like to note that this definition implies that for all $z_1, z_2 \in A$ we have $\delta_s(z_1 z_2^{p^{s+1}}) = \delta_s(z_1) z_2^{p^{s+1}}$.

Now we are able to give the required formula for $D_i(z)$ for $z \in A$. Namely let us represent $i = i_0 + i_1 p + \ldots + i_r p^r$ where all i_j, $0 \leqslant j \leqslant r$, are integers such that $0 \leqslant i_j < p$ and $i_r \neq 0$. Then we have

$$D_i(z) = \frac{1}{i_0! i_1! \ldots i_r!} \delta_0^{i_0} \delta_1^{i_1} \ldots \delta_r^{i_r}(z), \tag{1}$$

where $\delta_j^{i_j}$ is the i_j-th power of the operator δ_j (of course $\delta_j^0 = \mathrm{id}$ is the identity operator) for every j.

One can prove here the following fact. Let $f = f_1 f_2$ where $f_1, f_2 \in k_s[[X]][Y]$, $\deg_Y f_1 = n_1 \geqslant 1$. Put $A_1 = k_s((X))[Y]/(f_1)$ and $y_1 = Y \bmod f_1 \in A_1$. For all $z_1 \in A_1$ and integers $i, s \geqslant 0$ the elements $\delta_s(z_1), D_i(z_1) \in A_1$ are defined similarly to $\delta_s(z), D_i(z) \in A$ (see above; with k_s, f_1 in place of k, f). We have an epimorphism of $k_s((X))$–algebras $\psi : k_s \otimes_k A \to A_1$, $y \mapsto y_1$. Now we claim that $\psi(\delta_s(z)) = \delta_s(\psi(z))$, $\psi(D_i(z)) = D_i(\psi(z))$ for every $z \in A$ and for all $s, i \geqslant 0$.

The formula (1) has a theoretical importance. In practice to compute and estimate $D_i y$ it is better to proceed as follows. Assume additionally that $f \in k[X, Y]$. One finds the least integer s such that $p^s > i$, for all $0 \leqslant j < n$ one represents $y^{p^s j} = \sum_{0 \leqslant m < n} b_{j,m} y^m$ where all $b_{j,m} \in k(X)$ and solving a linear system over the field $k(X)$ finds the relation $y = \sum_{0 \leqslant j < n} a_j y^{p^s j}$ where all $a_j \in k(X)$. After that one represents $a_j = c_j / c^{p^s}$ where all $c_j, c \in k[X]$ and have the least possible degrees. Now $y = \sum_{0 \leqslant j < n} c_j Y^{p^s j} / c^{p^s}$ and $D_i y = \sum_{0 \leqslant j < n} D_i(c_j) \cdot Y^{p^s j} / c^{p^s}$. All $D_i(c_j)$ are easily computed since we know $D_i(X^m)$, see above.

At present we have an immediate application to the considered above problems. Namely, assume that the polynomial $f \in k[X, Y]$ has a root $Y = y_1 \in k_s[[X]]$. Put $f_1 = Y - y_1$. So f_1 divides f. Let us define w_1 similarly to w (with k_s, f_1 in place of k, f). Then one can easily see that $w_1|_{X=0} = y_1|_{X:=Z}$. Hence $(w_1|_{X=0})|_{Z:=X} = y_1$. Put $D_i y_1(0) = D_i y_1|_{X=0}$ for every i. Hence $y_1 = \sum_{i \geqslant 0} D_i y_1(0) X^i$. One can compute $D_i y$ as described above and after that find the minimal polynomial $\Phi_i \in k[X, Z]$ of the element $D_i y$ over $k(X)$ (we assume that X does not divide Φ_i). Also one can obtain good upper bounds for the sizes of coefficients from k of this polynomial Φ_i. Since $\psi(D_i y) = D_i y_1$ the element $D_i y_1$ is a root of the polynomial Φ_i. Hence $D_i y_1(0)$ is a root of the polynomial $\Phi_i(0, Y)$. So one can get the required efficient upper bound for the size of $D_i y_1(0)$.

Finally we would like to note that now in the case of zero characteristic applying the inverse function theorem (similarly to how it was above in nonzero characteristic) we can obtain the results of [8] by another method. This is also of great interest.

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Constructive tasks in distance Olympiads

Sergei Pozdniakov and Anton Chukhnov

Abstract. Constructive problems appear in every branch of mathematics. Constructive tasks are very important as they give us the direct way of stimulation of productive activity. This work is devoted to the position of contructive tasks in distance Olympiads on mathemathics and informatics. Also the tasks of this kind could be used for monitoring of students activity inside the learning process.

1. Constructive tasks

There are some general types of constructive tasks in mathematics. First of all there are explicit constructive tasks where the formulation of the task directly suggests to build a certain constrution.

Also the constructive task could hide behind the words "Does it exist...". The tasks of this type look more interesting but there is a rather big room for disappointment of schoolchildren losing hours for attempts to prove the negative answer instead of creating an example for the positive answer which sometines could be very simple.

Also there are "estimate plus example" tasks which require not only building the construction which is optimal for a certain parameter, but a proof of its optimality as well.

Building a construction is a productive activity. It could be more interesting for students than submitting numbers in formulas and more developing for them as well.

2. Constructive tasks in Olympiads

In the modern circumstances many Olympiads transformed some of their stages into the distant form. Within this form the organizerz face the choice how the participants should submit their solutions. The most common and the most simple way is to let the students just submit the answers. Unfortunately, sometimes it

is easy to guess the correct answer without doing all the necessary mental work needed to solve the task completely.

The second possible way is to have the full solution texts required. The minuses of this way are the low typewriting speed of some participants and troubles with submitting plots and figures.

Another possible way, which allows us to avoid a roulette from the one hand, and doesn't force the participants to type huge amount of text from another hand is using of constructive tasks. Of course, for each task or a type of task we have to develop a framework, which we call a "manipulator" and it requires some more activity than just checking a number answer, but it is the price for making the Olympiad more interesting and its results more representative. From the other hand, constructive task solutions can be verified automatically which helps us to reduce a human work comparing with the second way.

3. Constructive tasks within the framework of Olympiad in Discrete Mathematics and Theoretical Informatics

Within the framework of Olympiad in Discrete Mathematics and Theoretical Informatics we generally use six types of constructive tasks. Each of them is supported by its own manipulator.

We have logical schemes, Turing machines, finite state machines, regular expressions and graph manipulators and also the "Tarski World" manipulator which supports predicata calculus tasks.

For example, graph task may be formulated as "Find the minimal graph satisfying the certain conditions". Correctness of the constructed graph is verified automatically. The student can gain additional points for proving the minimality in the text form.

4. Using constructive tasks for non-invasive monitoring

While teaching students in the University you often ask yourself a question "how can i fairly access the students knowledges and undestanding". Exam is the necessary but not the best way just because in the stress situation some students show more than they can, and some other very much less. We may say that sometimes we check not the level of knowledge, but the ability to pass en exam.

This is the reason why we should develop non-invasive tools of assessment the students activity. Non-invasive means that while introducing this tools we don't force the students to prepare to pass the certain tests instead of learning the course as a whole.

This goal can be reached if we do not include our tools in the final assessment. So the only motivation for the student to solve those additional tasks would be the ability to check oneself and to improve ones knowledge.

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Relativistic Ovsyannikov vortex

A. P. Chupakhin and A. A. Yanchenko

Abstract. An exact solution of the Euler equations governing the flow of a compressible fluid in relativistic hydrodynamics is found and studied. It is a relativistic analogue of the Ovsyannikov vortex (special vortex) investigated earlier for classical gas dynamics. Solutions are partially invariant of Defect 1 and Rank 2 with respect to the rotation group. A theorem on the representation of the factor-system in the form of a union of a non-invariant subsystem for the function determining the deviation of the velocity vector from the meridian, and invariant subsystem for determination of thermodynamic parameters, the Lorentz factor and the radial velocity component is proved. Compatibility conditions for the overdetermined non-invariant subsystem are obtained.

Introduction

Equation of relativistic hydrodynamics governing the flow of a compressible fluid (gas dynamics) have the form [1]

$$\begin{aligned} &(\Gamma\rho)_t + \nabla\cdot(\Gamma\rho\vec{u}) = 0,\\ &(\Gamma^2\rho w\vec{u})_t + \nabla\cdot(\Gamma^2\rho w\vec{u}\times\vec{u}) + \nabla p = 0,\\ &(\Gamma^2\rho w - p)_t + \nabla\cdot(\Gamma^2\rho w\vec{u}) = 0. \end{aligned} \tag{1}$$

In (1) vector $\vec{u}$ — gas velocity in laboratory coordinate system, p, ρ — pressure and density, $w = 1 + \frac{\gamma}{\gamma-1}p/\rho$ — enthalpy, $\Gamma = (1-|\vec{u}|^2)^{-1/2}$ — Lorentz factor are function of independent variables — the time t and the spatial coordinates $\vec{x} = (x^1, x^2, x^3)$. A coordinate system is chosen such that the speed of light $c = 1$ and for the velocity modulus $|\vec{u}| = (\sum_{i=1}^{3} {u^i}^2)^{1/2}$ inequality $|\vec{u}| < 1$ is satisfied.

Lemma 1. The Euler equations of compressible fluid for relativistic hydrodynamics can be written in the form

$$\begin{aligned} &b\,\mathrm{D}\vec{u} + p_t\,\vec{u} + \nabla p = 0,\\ &\mathrm{D}a + a\,\mathrm{div}\,\vec{u} = 0,\\ &a\,\mathrm{D}d = p_t. \end{aligned} \tag{2}$$

In (2) $a = \Gamma\rho$, $b = \Gamma^2\rho w$, $d = b/a = \Gamma w$, $D = \partial_t + \vec{u}\cdot\nabla$ — total derivative.

We introduce spherical coordinates (r, θ, ϕ), (U, V, W) and new coordinates for tangent component $\vec{u}_\tau = (V, W)$ of the velocity vector:

$$V = H\cos\omega, \quad W = H\sin\omega, \tag{3}$$

such that $H = \sqrt{V^2 + W^2}$, $\omega = \arctan W/V$. Special vortex is a solution of equations (2) in which a special dependence of the functions on the independent variables is realized, namely:

$$\begin{aligned} &U = U(r,t), \quad \Gamma = \Gamma(r,t), \quad H = H(r,t), \\ &p = p(r,t), \quad \rho = \rho(r,t), \quad w = w(r,t), \\ &\omega = \omega(t, r, \theta, \varphi). \end{aligned} \tag{4}$$

From the general theory of group analysis of differential equations [3], after substitution of the representation (4) in (2), we obtain composition of a factor-system in the form of a union of an overdetermined system of differential equations for function ω and a system of equations for invariant functions U, Γ, H, p, ρ, w.

Special vortex equations for relativistic gas dynamics

Lemma 2. Special vortex equations for relativistic gas dynamics are represented as a union of invariant subsystem

$$\begin{aligned} &ad(\mathrm{D}_0 U - \frac{1}{r}H^2) + p_t U + p_r = 0, \\ &\mathrm{D}_0 H + \frac{U}{r}H + p_t H = 0, \\ &a\mathrm{D}_0 d = p_t, \end{aligned} \tag{5}$$

where $\mathrm{D}_0 = \partial_t + U\partial_r$ — invariant part of the total derivative, and overdetermined subsystem for function ω:

$$\begin{aligned} &k\sin\theta \mathrm{D}_0\omega + \sin\theta\cos\omega\omega_\theta + \sin\omega\omega_\varphi + \cos\theta\sin\omega = 0, \\ &\sin\theta\sin\omega\omega_\theta - \cos\omega\omega_\varphi = h\sin\theta + \cos\theta\cos\omega, \end{aligned} \tag{6}$$

where

$$k = r/H, \quad h = k(a^{-1}\mathrm{D}_0 a + r^{-2}(r^2 U)_r). \tag{7}$$

It is remarkable that (6) and (7) coincide exactly with the corresponding equations for classical gas dynamics [2, 4]. But function h is different, it is related to other physical quantities.

Lemma 3. (Ovsyannikov compatibility condition) Compatibility condition of overdetermined system (6) has the form

$$k\mathrm{D}_0 h = h^2 + 1. \tag{8}$$

Equations (8) complement invariant subsystem (5). Thus, all mathematical results proved for the special vortex in classical gas dynamics [2, 4] are carried over to special vortex in relativistic gas dynamics.

Stationary special vortex for relativistic gas dynamics

Consider a stationary special vortex, i.e. solution which is partially invariant with respect to the group $\langle \partial_t, SO(3) \rangle$. In representation (4) and in equations of Lemma 2, it is necessary to remove the dependence on time, then $\mathrm{D}_0 = U\frac{d}{dr}$. Consider monatomic gas with $\gamma = 5/3$.

Lemma 4. For stationary special vortex for relativistic gas dynamics invariant subsystem is reduced to ordinary implicit differential equation

$$F(R,h,p;m_0,s_0) \equiv q^{3/2} - R^2 p\Big(3m_0 + s_0\frac{p^2}{1+h^2}\Big)q + 3m_0^2R^4p^2q^{1/2} - m_0^3R^6p^3 = 0, \tag{9}$$

where $p = dh/dR$; $s_0 > 0$, $0 < m_0 < 1$ — constants characterizing physics of the problem, $R > 1$ — normalized distance, $q = q(R,h,p)$ has the form

$$q(R,h,p) = R^2(R^2-1)p^2 - (1+h^2)^2. \tag{10}$$

Thus, the determination of the special vortex for relativistic gas dynamics reduces to solving equation (9) and overdetermined system (6). This system is integrated in finite form in [2], its solution ω is given by an implicit function. Geometric interpretation and domain investigation are given in [5].

Equation (9) belongs to a class of equations that are unresolved with respect to derivatives. Now such equations are usually called implicit differential equations. The present state of the theory is presented in [6]. The specificity of equations of this type is the existence of manifold of branching solutions, the presence of trajectories bundle starting from singular points of different degrees of degeneracy [6].

Investigation of singular points of equation (9)

Implicit equation (9) can be resolved with respect to derivative p at all points of $\mathbb{R}^3(R,h,p)$, except for the points of manifold

$$F(R,h,p) = 0, \quad F_p(R,h,p) = 0. \tag{11}$$

Curve (11) is called a criminant of equation (9), it is manifold of branching of integral curves. It consists of singular points of equation (9), called regular singular points. Folded singular points can be found from the system of equations

$$F(R,h,p) = 0, \quad F_p(R,h,p) = 0, \quad F_R(R,h,p) + pF_h(R,h,p) = 0. \tag{12}$$

Projection of the criminant curve onto plane $\mathbb{R}^2(h,R)$ is called a discriminant curve.

The essence of the geometric approach to study of implicit differential equations, proposed by Poincaré, is to raise the equation to a vector field

$$R_\tau = F_p, \quad h_\tau = pF_p, \quad p_\tau = -(F_R + pF_h), \tag{13}$$

where τ — new parameter along integral curve. With this interpretation, the integral curves of equation (9) are located on different sheets of surface $F = 0$. These integral curves can be projected onto the plane $\mathbb{R}^2(h,R)$ with overlapping.

The investigation of the singular points of equations (9) is connected with large computational difficulties. The solution of the system of equations (12), determined by the resultants of the corresponding polynomials, can be obtained by means of a system of symbolic and numerical computations. Even with the use of such computing systems, the calculation can be quite long.

Numerical techniques

We will deal with F^2 because it is more convenient from the computational point of view (everywhere below by F we mean F^2).

Discriminant curve

The discriminant curve is given by the system of equations (11) To solve this and other similar systems, the following technique is used. Since the equations of the system (11) are polynomials in p, we can eliminate p by constructing the resultant of two these polynomials. Denote by $\Phi_1 = F$, $\Phi_2 = F_p$. Then

$$R_{12}(R,h) = \mathrm{res}(\Phi_1, \Phi_2). \tag{14}$$

Further, in the (R,h)-plane, we can numerically solve the algebraic equation $R_{12}(R,h) = 0$ for h. Thus, for given m_0, s_0 and R, we can calculate the discriminant curve.

Theorem 1 (about discriminant curve) The discriminant curve consists of one or two components. The discriminant curve always has its "main" component, but for small s_0 ($s_0 \approx 10^{-2}$ and less) an additional ("secondary") component of the discriminant curve appears.

Folded singular points

Folded singular points are found from the system of equations (12). Denote by $\Phi_1 = F$, $\Phi_2 = F_p$, $\Phi_3 = F_R + pF_h$. In order to solve this system, we construct the following resultants

$$R_{12}(R,h) = \mathrm{res}(\Phi_1, \Phi_2),\ \ R_{13}(R,h) = \mathrm{res}(\Phi_1, \Phi_3). \tag{15}$$

Intersection of curves $R_{12} = 0$ and $R_{13} = 0$ gives us a set of folded singular points. To find $R_{12}(R,h)$, it is necessary to calculate determinant of order 19 ($11! = 39{,}916{,}800$ terms, without zeros). After simplification, the degree of the polynomial obtained is 60 and 108 for R and h, respectively. To find $R_{13}(R,h)$ we calculate determinant of order 20 ($15! = 1{,}307{,}674{,}368{,}000$ terms, without zeros). The degree of $R_{13}(R,h)$ is 64 for R and 170 for h.

As a result of the numerical experiment, the following statement was obtained.

Theorem 2 (existence and uniqueness of folded singular points) For any values of parameters s_0 and m_0 there is a unique folded singular point.

Conclusion

A stationary solution of relativistic Ovsyannikov vortex is studied in detail. It is proved that its invariant subsystem reduces to an implicit differential equation. For this equation, the manifold of branching of solutions is investigated, and a set of singular points is found.

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Tropical Orthogonal Linear Prevarieties

Dima Grigoriev (jointly with Nikolai Vorobjov)

We study the operation $A^{\perp}$ of tropical orthogonalization, applied to a subset A of a vector space $(\mathbf{R} \cup \{\infty\})^n$, and iterations of this operation. Main results include a criterion and an algorithm, deciding whether a tropical linear *prevariety* is a tropical linear *variety*, formulated in terms of a duality between $A^{\perp}$ and $A^{\perp\perp}$. We give an example of a countable family of tropical hyperplanes such that their intersection is not a tropical prevariety. We design an auxiliary algorithm, with singly exponential complexity, which for a given algebraic set $V \subset (\mathbf{C}((t^{1/\infty})))^n$ and a point $\mathbf{u} \in \mathbf{Q}^n$ decides whether or not $\mathbf{u}$ belongs to the tropicalization of V, and, if it does, produces a lifting of $\mathbf{u}$ in V. An algorithm for the same problem due to A. Jensen, H. Markwig, T. Markwig has a doubly exponential complexity bound in the number of variables

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Landau: language for dynamical systems with automatic differentiation

Ivan Dolgakov and Dmitry Pavlov

Abstract. Most numerical solvers used to determine parameters of dynamical systems rely on first-order derivatives of the state of the system w. r. t. the parameters. The number of parameters can be fairly large. One of the approaches of obtaining those derivatives is the integration of the derivatives simultaneously with the dynamical equations, which is best done with the automatic differentiation technique.
Even though there are known some automatic differentiation tools, there is no framework providing the solution fast and useful enough for dynamic system modeling purposes. Landau is the Turing incomplete statically typed domain-specific language aimed to fill this gap. The Turing incompleteness provides an ability of sophisticated source code analysis and as a result a highly optimized compiled code. Among other things the language syntax supports functions, compile-time ranged for loops, if/else branching constructions, real variables, and arrays, ability to manually discard calculation where the automatic derivatives values are expected to be negligibly small. In spite of reasonable restrictions, the language is rich enough to express and differentiate any cumbersome paper-equation with practically no effort.

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RSK bumping trees and a fast RSK algorithm

Vasilii Duzhin, Artem Kuzmin and Nikolay Vassiliev

The Robinson-Schensted-Knuth (RSK) correspondence is a bijection between a set of permutations of integers and a set of pairs of Young tableaux of the same shape: insertion tableaux P and recording tableaux Q. The procedure of transforming the input permutation into tableaux is also known as the RSK algorithm or the RSK transformation. The RSK algorithm has many important applications in combinatorics and representation theory.

Each number from the input permutation is being put into a certain place of tableau P, consequently displacing other numbers when it is necessary. A *bumping route* [1] is a sequence of positions in insertion tableau where bumping occures during the RSK transformation. At the same time, tableau Q "records" the position where the form of P has changed by putting the index of the current number at the same position. A more detailed description of the RSK algorithm can be found in [2].

The RSK algorithm can be easily generalized to the infinite case: the input infinite sequence of numbers can be transformed into a pair of infinite Young tableaux. Some results of numerical experiments using RSK transformation of extremely large input sequences are shown in [3]. For such experiments, the efficiency of the algorithm becomes especially crucial.

The goal of this work is to implement a special variant of RSK which works significantly faster than the original algorithm. The computational costs of the RSK are mainly caused by searching a position where the next number should be bumped in tableau P. In order to solve this problem, we consider tableau P together with a special combinatorial object called a *bumping forest* which is a union of all possible bumping routes of an insertion tableau. The Figure 1 (a) shows an example of a Young tableau equipped with a bumping forest.

The bumping forest itself is shown in Fig. 1 (b). It can be easily seen that it consists of connected components, which we call *bumping trees*. Each bumping tree is a union of bumping routes converging to a same position where the form of tableaux P,Q can be changed.

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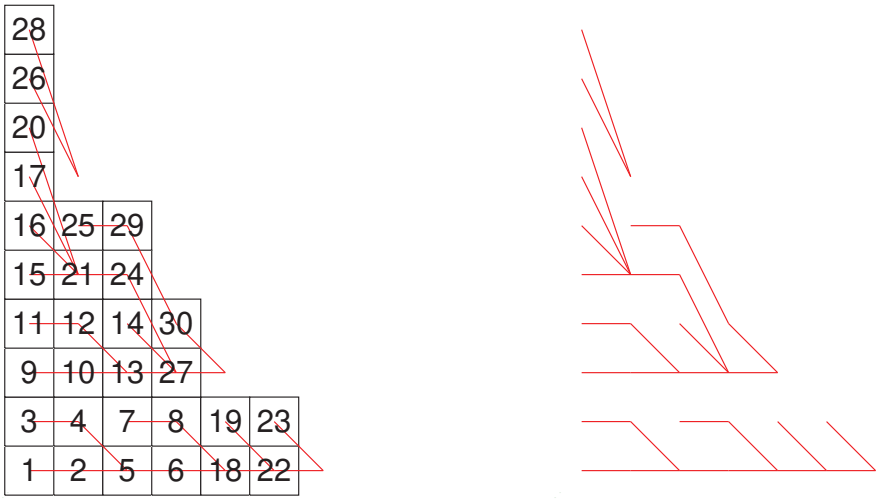

FIGURE 1. A Young tableau and its bumping forest

At each step of the algorithm we need to maintain the correct structure of a bumping forest. This maintenance expenses costs us some computational resources. On the other hand, we do not need to calculate bumping routes for the input numbers. The numerical experiments show that the proposed algorithm works faster than the standard RSK algorithm and the performance gain is higher for larger permutations. Table 1 shows the calculation time of both standard and proposed versions of RSK for different uniformly-distributed random permutations of integers. Each value is an average elapsed time of processing 300 permutations of the same size.

TABLE 1. The comparison between the speed of standard and fast RSK algorithms

Permutation size	Elapsed time by standard RSK (in sec)	Elapsed time by fast RSK (in sec)
100000	5	1
250000	22	2
500000	72	5
1000000	274	15

As we can see from the table, the proposed algorithm works ≈ 18 times faster than the standard one for the permutations of size 10^6.

Note that the fast inverse RSK transformation can be implemented using the bumping forest as well. In that case the bumping routes will be reversed.

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Calculation of High Orders of the Resonance Normal Form

Victor F. Edneral

Abstract. It is well known that the most interesting properties of ODEs system are mainly defined by the lowest no vanished orders of the normal form. Unfortunately, such an order can be very high.

We study integrability of the degenerated 5 parametric planar ODEs system [1] by the normal form method [2, 3] where the resonant normal form is calculated by the NORT program [4].

We have found six first integrals of motion of the system at different sets of the parameters. But the last step of the study demands a calculation of the corresponding normal form till the 27^{th} order. I.e. it needs to calculate the truncated power series in two small variables and 4 no small parameters until the 27^{th} order. This problem does not solve yet.

In the report, we discuss the internal representation of series in the STANDARD LISP program NORT and give estimations for a calculation of the problem by a straight method and with the usage of lazy and modular calculations.

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The Gamma function and its inverse

A. C. Camargos Couto and D. J. Jeffrey

Abstract. We review some properties of the Gamma function of Euler, $\Gamma(z)$, paying particular attention to the complex plane. We than define the inverse Gamma function, $\overset{\vee}{\Gamma}_k(z)$, which is a multi-valued function, and therefore must have its branches defined for all complex values. The branches are defined by considering first the range of $\overset{\vee}{\Gamma}$, the domain of Γ, and then transforming to the domain of $\overset{\vee}{\Gamma}$, the range of Γ. Having delineated all branches, we then present numerical algorithms for the evaluation of $\overset{\vee}{\Gamma}$.

1. Introduction

A recent review [1] of the factorial or Γ function noticed some interesting facts. The asymptotic approximation known as Stirling's formula was really due to de Moivre, and Stirling's own formula is actually more accurate, in some sense. The two approximations are respectively

$$\ln\Gamma(z) = \tfrac{1}{2}\ln 2\pi + (z-\tfrac{1}{2})\ln z - z + \textstyle\sum_{n=1}^{\infty} \frac{\mathcal{B}(2n)}{2n(2n-1)z^{2n-1}} \ ,$$

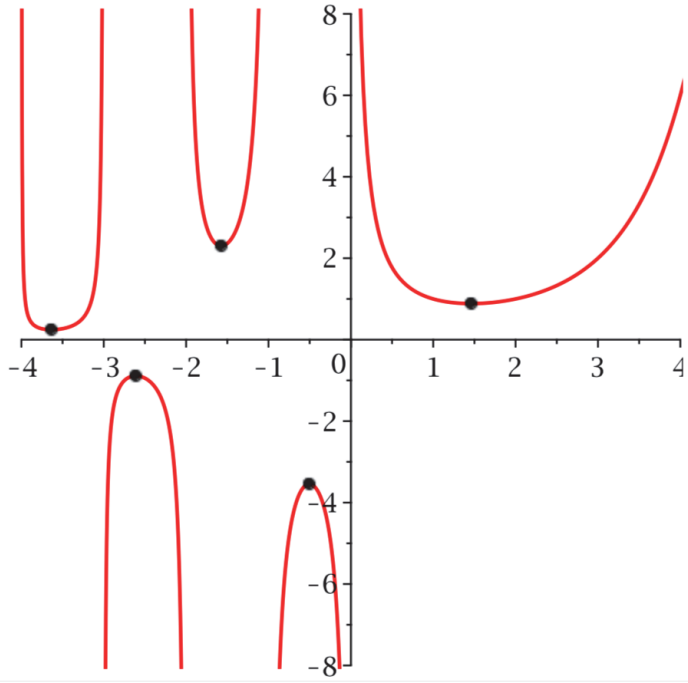
where $\mathcal{B}$ is a Bernoulli number. This form was due to de Moivre. Stirling himself derived

$$Z = z - \tfrac{1}{2} \ ,$$
$$\ln\Gamma(z) = \tfrac{1}{2}\ln 2\pi + Z\ln Z - Z - \textstyle\sum_{n=1}^{\infty} \frac{(1-2^{1-2n})\mathcal{B}(2n)}{2n(2n-1)Z^{2n-1}} \ .$$

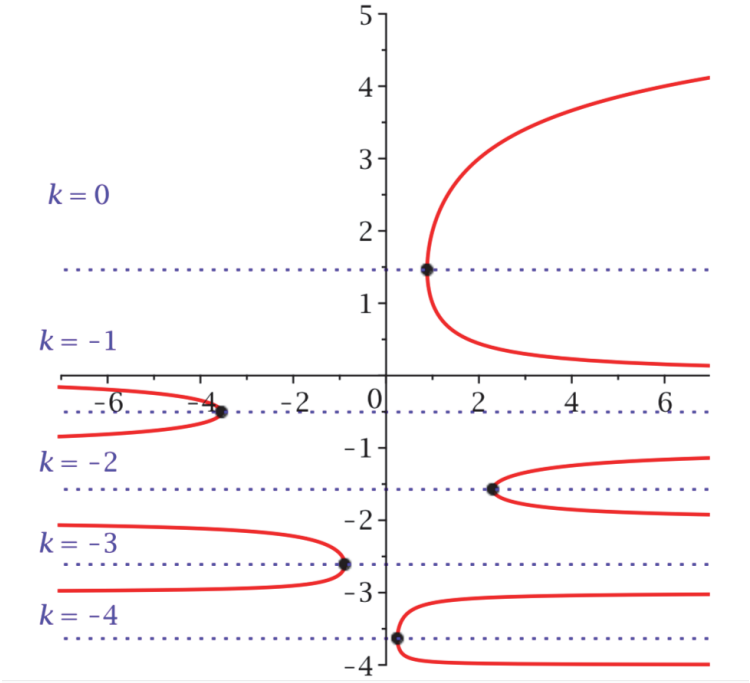
Derivations of Stirling's formula are very popular amongst contributors to journals such as the American Math. Monthly: about 30 variations on the derivation have been published. For many authors, the derivation of the formula is more important than its utility. In fact we shall show it is remarkably accurate even in the complex domain.

Another observation was that only a few papers have addressed the question of the functional inverse of Γ. One of the earliest applications of inverse Γ was made by Gaston Gonnet in 1981 [2] (in the same paper that defined W). The inverse is a new challenge for the approach to understanding multi-valued inverse

functions in the complex plane presented in [3]. In order to visualize the branch cuts in the real line, take $\Gamma(x)$ over $-4 < x < 4$:



The points at which the branch cuts are defined should eliminate the multi-valued component from $\check{\Gamma}$ at the intervals constrained by the cutting lines. We therefore chose the local minima and maxima of $\Gamma(x)$ for the location of those points. $\check{\Gamma}$ is then treated as a single valued function at those intervals and labelled with a branch index k:



We shall describe work on evaluation on the real line and the discuss possible extension of branches in the complex plane.

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Sketch on quaternionic Lorentz transformations

Mikhail Kharinov

Abstract. Lorentz transformations are decomposed into a linear combination of two orthogonal transformations. In this way a two-term expression of Lorentz transformations by means of quaternions is proposed. An analytical solution to the problem of finding eigenvectors is given. The conditions for the existence of eigenvectors are specified. A quartet of eigenvectors that occurs when rotational axis is orthogonal to velocity direction is obtained. The accompanying relativistic velocity addition is discussed.

Introduction

W.R. Hamilton had discovered quaternions in the 19 century in order best to describe real four-dimensional spacetime $\mathbb{R}^4$, supplied with a *cross product* $[\nu, n]$ of spatial vectors ν and n. The main advantage of quaternions is that they allow working with linear transformations of 4-dimensional Euclidian space without explicitly introducing a standard orthonormal basis and matrix representation of a linear operator. The use of quaternion multiplication provides concise calculations. So, the rotation V of a 3-dimensional space is elegantly described through multiplication of quaternions as $V\{u\} = bu\bar{b}$, where: $b = i_0 \cos\frac{\varphi}{2} + \nu \sin\frac{\varphi}{2}$; $\bar{b} = i_0 \cos\frac{\varphi}{2} - \nu \sin\frac{\varphi}{2}$; i_0 is the multiplicative unity; ν is the *unit* vector of unit length along rotational axis, such that $\sqrt{(\nu,\nu)} = 1$ and φ is the rotational angle [1].

In the quaternion space, the Lorentz transformations are expressed only slightly more complicated than the rotation V.

1. Lorentz transformations in terms of quaternions

The Lorentz transformations $\mathcal{L}$ are defined as a linear transformation of the space of quaternions u, v that preserves the real inner product $(u, \bar{v})$ of one *conjugated* vector $\bar{v} = 2(v, i_0) - v$ by another vector u: $(\mathcal{L}\{u\}, \overline{\mathcal{L}\{v\}}) = (u, \bar{v})$. The Lorentz transformations $\mathcal{L}$ is decomposed into the two simple transformations V and L as in [2], so that $\mathcal{L}\{u\} = \pm VL\{u\}$ or $\mathcal{L}\{\bar{u}\} = \pm VL\{\bar{u}\}$. For brevity, only one option

$\mathcal{L}\{u\} = VL\{u\}$ is treated, where L is the Lorentz boost. It is *screw-symmetrical*: $(L\{u\}, v) = (L\{v\}, u)$ for any u, v.

$L\{u\}$ is expressed in quaternions as:

$$L\{u\} = \bar{a}ua - n\bar{u}\sinh\theta \equiv au\bar{a} - \bar{u}n\sinh\theta, \tag{1}$$

where $a = i_0\cosh\frac{\theta}{2} + n\sinh\frac{\theta}{2}$ and θ is the *rapidity*, such that the velocity vector $\mathbf{v}$ divided by scalar speed of light $\mathbf{c}$ is expressed as $\mathbf{v}/\mathbf{c} = n\tanh\theta$.

It is noteworthy that in [3, 4] the Lorentz boost is described by the truncated formula (1). But this is achieved only due to extra dimensionality, which complicates interpretation.

The dual expression (1) for L and simple quaternion multiplication rules [1, 5] provide easy operation with the Lorentz transformations $\mathcal{L} = VL$ in a coordinate-free way. It is a good exercise to obtain eigenvectors c_k for the transformation $\mathcal{L} = VL$: $\mathcal{L}\{c_k\} = \xi_k c_k$, where ξ_k are real eigenvalues and eigenvector sequence number k starts from 0 and does not exceed 3.

2. Eigenvectors in the general case

It is trivial that the eigenvectors of the transformation $\mathcal{L} = VL$, corresponding to eigenvalues other than 1, are *pseudo-orthogonal* to themselves and to the *invariant* eigenvectors that correspond to $\xi = 1$, e.g. for $\xi_0 \neq 1, \xi_1 \neq 1, \xi_3 = \xi_4 = 1$ the formulae $(c_0, \bar{c}_0) = (c_1, \bar{c}_1) = (c_0, \bar{c}_2) = (c_0, \bar{c}_3) = (c_1, \bar{c}_2) = (c_1, \bar{c}_3) = 0$ are valid. A concomitant fact is that the eigenvalues are pairwise mutually inverse due to the invariance of the equation for ξ with respect to the replacement of ξ by ξ^{-1}.

For easy finding of real eigenvalues, it is convenient to present the general equation for ξ as follows:

$$(\xi - \xi_0)(\xi - \xi_0^{-1})[\xi^2 + \xi(2x - \alpha - \beta) + 1] = 0, \tag{2}$$

where x, which must be outside the interval $(0, 1)$, is found from the equation:

$$x^2 - x\frac{\alpha + \beta}{2} + \frac{\alpha - \beta}{2} - 1 = 0, \tag{3}$$

ξ_0 is found from the equation

$$\frac{\xi_0 + \xi_0^{-1}}{2} = x \tag{4}$$

and $\alpha = (\cosh\theta + 1)(1 + \cos\varphi) \geq 0, \beta = (\nu, n)^2(\cosh\theta - 1)(1 - \cos\varphi) \geq 0$. Concerning the latters it should be noted that in the expressions for α and β, the values of θ and φ are assumed to be non-trivial and *variative*, i.e. both are not fixed for given rotational axis ν and velocity direction n.

With positive α and β the equation (3) for x has at least one required solution $x > 1$. In this case, a pair of mutually inverse eigenvalues other than 1 is available. For each eigenvalue $\xi \neq 1$ the corresponding eigenvector is represented as $i_0 - d$,

where $(d, i_0) = 0$, $(d, d) = 1$. In turn, the components of the vector d are calculated by the formulae:

$$(d, \nu) = (\nu, n)\frac{\cosh\theta - 1}{\sinh\theta}\frac{\xi + 1}{\xi - 1},\ (d, n) = \frac{\xi - \cosh\theta}{\sinh\theta},$$
$$(d, [\nu, n]) = [(d, n) - (\nu, n)(d, \nu)]\frac{\xi\sin\varphi}{\xi\cos\varphi - 1}. \tag{5}$$

3. The case of velocity, orthogonal to the rotational axis

All four eigenvectors are available in the special case of $(\nu, n) = \beta = 0$. In this case, the equation for x becomes trivial:

$$(x - 1)\left(x + 1 - \frac{\alpha}{2}\right) = 0. \tag{6}$$

In the case of $x = \frac{\alpha}{2} - 1$ the equation for finding ξ_0 and $\xi_1 = \xi_0^{-1}$ is expressed by the formula:

$$\xi^2 - \xi(\alpha - 2) + 1 = 0. \tag{7}$$

As follows from the last formula (7), in order to successfully find the target values of ξ_0 and $\xi_1 = \xi_0^{-1}$, the next condition must be satisfied:

$$\left|\sin\frac{\varphi}{2}\right| \le \left|\tanh\frac{\theta}{2}\right| \Leftrightarrow \left|\cos\frac{\varphi}{2}\right|\cosh\frac{\theta}{2} \ge 1 \Leftrightarrow \left|\tan\frac{\varphi}{2}\right| \le \left|\sinh\frac{\theta}{2}\right|, \tag{8}$$

where pair vertical lines denotes absolute value. In the case of $x = 1$, we get the trivial equation for ξ: $(\xi - 1)^2 = 0$ and obtain a pair $\xi_3 = \xi_4 = 1$ of unit values of ξ corresponding to a pair of invariant eigenvectors.

Explicit expressions for eigenvectors and eigenvalues are listed in the table 1, wherein ξ_0 is the solution of (7) under the condition (8).

Notation	Eigenvector	Eigenvalue
c_0	$i_0 - \left(n + [\nu, n]\frac{\xi_0\sin\varphi}{\xi_0\cos\varphi - 1}\right)\frac{\xi_0 - \cosh\theta}{\sinh\theta}$	ξ_0
c_1	$i_0 - \left(n + [\nu, n]\frac{\sin\varphi}{\cos\varphi - \xi_0}\right)\frac{1 - \xi_0\cosh\theta}{\xi_0\sinh\theta}$	ξ_0^{-1}
c_2	$i_0 + \left(n - [\nu, n]\cot\frac{\varphi}{2}\right)\tanh\frac{\theta}{2}$	1
c_3	ν	1

TABLE 1. Eigenvector quartet in the case of $(\nu, n) = 0$

Any vector u is trivially decomposed into a linear combination of the listed eigenvectors:

$$u = \frac{c_0(u, \bar{c}_1) + c_1(u, \bar{c}_0)}{(c_0, \bar{c}_1)} + c_2\frac{(u, \bar{c}_2)}{(c_2, \bar{c}_2)} + c_3\frac{(u, \bar{c}_3)}{(c_3, \bar{c}_3)}. \tag{9}$$

Note that the expansion (9) is available only if the condition (8) is fulfilled.

Conclusion

Case $(\nu, n) = 0$ is the most important because it is this case that arises in relativistic addition of velocities, interpreted in terms of Lobachevsky theory [6, 7, 8]. However, according to the authoritative opinion of John Frederick Barrett, "The hyperbolic theory is not at all new and was described by V. Varicak shortly after Einstein's initial work. But it has been ignored now for over 100 years by the mainstream theory." Perhaps, the task of obtaining of the eigenvectors for the Lorentz transformations represented in quaternions, and also in octonions, will be useful for further development in this direction.

In the following papers it will be shown that the decomposition (9) of any vector into eigenvectors is not available for relativistic addition of velocities. Perhaps, professional physicists will give a plausible interpretation for this. In any case, the quaternion technique of working with spatial transformations seems useful for solving modern engineering problems.

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Invariant Projectors in Wreath Product Representations

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Abstract. An algorithm for computing the complete set of irreducible invariant projectors in the space of the permutation representation of a wreath product is described. This set provides the irreducible decomposition of the representation. The corresponding C program constructs decompositions of representations of high dimensions and high ranks.

1. Introduction

A description of a physical system commonly involves a *space* X, on which a group of *spatial symmetries* G (or $G(X)$) acts, and a set of *local states* V with a group of *local symmetries* F (or $F(V)$). X, V and F can be treated, respectively, as the *base*, the *typical fiber* and the *structure group* of a *fiber bundle*. A state of the whole system is a function from X to V, i.e., a *section* of the bundle. A natural symmetry group that acts on the set of sections V^X and preserves the structure of the bundle is the *wreath product* [1, 2] of F and G

$$\widetilde{W} = F \wr G \cong F^X \rtimes G\,. \tag{1}$$

The action of $\widetilde{W}$ on V^X is defined by

$$v(x)\,(f(x)\,,g) = v\big(xg^{-1}\big)\,f\big(xg^{-1}\big)\,,$$

where $v \in V^X$, $f \in F^X$, $g \in G$; the *right-action* convention is used for all group actions. Importance of wreath products:

1. The *universal embedding theorem* (Kaloujnine-Krasner) states that any extension of group A by group B is isomorphic to a subgroup of $A \wr B$, i.e., the wreath product is a universal object containing all extensions.
2. Classification of *maximal subgroups* of the *symmetric* group (the *O'Nan-Scott theorem*) essentially involves wreath products [3].
3. The wreath product $\mathsf{S}_m \wr \mathsf{S}_n$ is the automorphism group of the *hypercubic graph* or *Hamming scheme* $H(n,m)$ in coding theory [4].
4. Unitary representations of wreath products arise naturally in the study of multipartite quantum systems.

The main step in the study of group representations is to decompose them into irreducible components. Our algorithm [5] decomposes representations of finite groups via computing a *complete set of mutually orthogonal irreducible invariant projectors*. A similar construction in ring theory is called a *complete set of primitive*

orthogonal idempotents. An arbitrary ring with such a set can be represented as a direct sum of indecomposable rings. This is called a *Peirce decomposition* [6, 7]. In our case, *irreducible invariant projectors* are *primitive idempotents* of the *centralizer ring* of a group representation. The dimension of this ring is called the *rank of the representation*. The program in [5] proved to be very effective in problems with low ranks. In particular, it coped with many high dimensional representations of simple groups and their "small" extensions (which typically have low ranks), presented in the ATLAS [8]. However, wreath products, which contain all possible extensions, are far from simple groups and their representations have high ranks. The approach proposed here allows us to decompose wreath product representations with very high dimensions and ranks.

2. Centralizer Ring of Wreath Product Representation

We assume that $X \cong \{1,\dots,N\}$ and $V \cong \{1,\dots,M\}$, and hence $G(X) \leq \mathsf{S}_N$ and $F(V) \leq \mathsf{S}_M$. The permutation representation $\widetilde{P}$ of $\widetilde{W}$ is defined by $(0,1)$-matrices of the size $M^N \times M^N$ that have the form

$$\widetilde{P}(\widetilde{w})_{u,v} = \delta_{u\widetilde{w},v}, \text{ where } \widetilde{w} \in \widetilde{W};\ u,v \in V^X;\ \delta \text{ is the } \textit{Kronecker delta}.$$

As a representation space, we assume an M^N-dimensional Hilbert space $\widetilde{\mathcal{H}}$ over some abelian extension of $\mathbb{Q}$ being a splitting field for the local group F. We denote the rank of the representation $\widetilde{P}$ by $\widetilde{R}$, and we denote the basis of the centralizer ring by $\widetilde{A}_1,\dots,\widetilde{A}_{\widetilde{R}}$. The basis elements are solutions of the system of equations

$$\widetilde{P}\big(\widetilde{w}^{-1}\big)\,\widetilde{A}\widetilde{P}(\widetilde{w}) = \widetilde{A},\ \widetilde{w} \in \widetilde{W}\,. \tag{2}$$

A more detailed analysis of (2), taking into account the structure of the wreath product (1), allows to obtain explicit expressions for the basis elements of the centralizer ring of $\widetilde{P}$

$$\widetilde{A}_r = \sum_{q \in rG} A_{q_1} \otimes \cdots \otimes A_{q_N}\,. \tag{3}$$

Here

1. R and $A_1,\dots,A_R$ are, respectively, the rank and the basis of the centralizer ring for the M-dimensional permutation representation of the local group F.
2. $r \in \overline{R}^X$ denotes a mapping from X into $\overline{R} = \{1,\dots,R\}$.
3. rG is the G-orbit of the mapping $r \in \overline{R}^X$ with respect to the action defined by $rg = [r_{1g},\dots,r_{Ng}]$ for $g \in G$. The notation $r = [r_1,\dots,r_N]$ is assumed.

It is easy to verify that the basis elements (3) form a complete system, i.e.,

$$\sum_{i=1}^{\widetilde{R}} \widetilde{A}_{r^{(i)}} = \mathbb{J}_{M^N},$$

where $\mathbb{J}_{M^N}$ is the $M^N \times M^N$ *all-ones matrix*, $r^{(i)}$ denotes some numbering of the orbits of G on $\overline{R}^X$.

3. Complete Set of Irreducible Orthogonal Invariant Projectors

The complete set of irreducible orthogonal invariant projectors is a subset of the centralizer ring, specified by the conditions of idempotency and mutual orthogonality. Using the properties of the tensor (Kronecker) product [9], their consequences and some additional technical considerations we come to the following.

Let $B_1, \ldots, B_K$ be the complete set of irreducible orthogonal projectors in the permutation representation of the local group F. Let $\overline{K} = \{1, \ldots, K\}$ and $\overline{K}^X$ be the set of all mappings from X into $\overline{K}$. The action of $g \in G$ on the mapping $k \in \overline{K}^X$ is defined as $kg = [k_{1g}, \ldots, k_{Ng}]$. Then we have

Proposition. *The irreducible orthogonal invariant projector in the permutation representation of the wreath product takes the form*

$$\widetilde{B}_k = \sum_{\ell \in kG} B_{\ell_1} \otimes \cdots \otimes B_{\ell_N} \,, \tag{4}$$

where kG denotes the G-orbit of the mapping k on the set $\overline{K}^X$.

The easily verifiable completeness condition $\sum_{i=1}^{\widetilde{K}} \widetilde{B}_{k^{(i)}} = \mathbb{1}_{M^N}$ holds. Here $\widetilde{K}$ is the number of irreducible components of the wreath product representation, $\mathbb{1}_{M^N}$ is the identity matrix in the representation space, $k^{(i)}$ denotes some numbering of the orbits of G on $\overline{K}^X$.

To compute the basis elements (3) of the centralizer ring and projectors (4), we wrote a program in C. The input data for the program are the generators of the spatial and local groups, and the complete set of irreducible invariant projectors of the local group (obtained, for example, by the program described in [5]).

4. Calculation Example

We give here the calculation for the representation of the wreath product of the rotational symmetry groups of the octahedron and icosahedron. The dimension and rank are $M^N = 2\,176\,782\,336$ and $\widetilde{R} = 122\,776$.

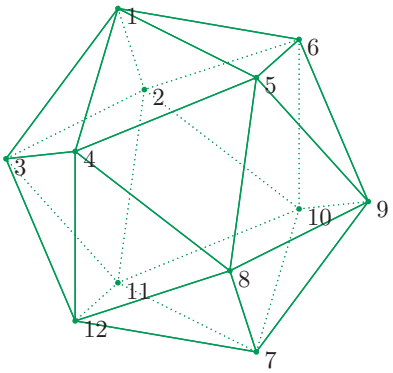

FIGURE 1. Icosahedron.

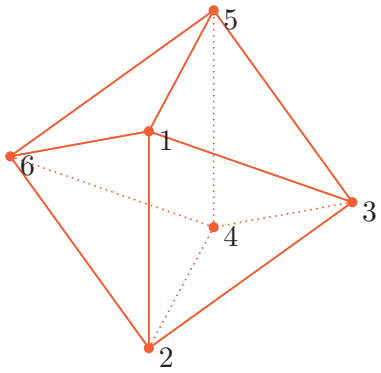

FIGURE 2. Octahedron.

Space Group. The space X is the icosahedron whose vertices form the set of points: $X \cong \{1,\dots,12\}$, see Fig. 1. As a group of spatial symmetries we take the group A_5, which describes the *rotational* (or *chiral*) *symmetries* of the icosahedron. For the vertex numbering as in Fig. 1, the space symmetry group can be generated by two permutations:

$$G(X) = \big\langle (1,7)(2,8)(3,12)(4,11)(5,10)(6,9),\ (2,3,4,5,6)(8,9,10,11,12) \big\rangle \cong \mathsf{A}_5.$$

Local Group. The local states, $V \cong \{1,\dots,6\}$, are the vertices of the octahedron. The group of rotational symmetries of the octahedron is S_4. For the vertex numbering of Fig. 2, the local symmetry group has the following presentation by two generators

$$F(V) = \big\langle (1,3,5)\,(2,4,6)\,,\ (1,2,4,5) \big\rangle \cong \mathsf{S}_4.$$

The six-dimensional permutation representation $\underline{\mathbf{6}}$ of $F(V)$ has rank 3, and the basis of the centralizer ring is

$$A_1 = \mathbb{1}_6,\ A_2 = \begin{pmatrix} \mathbb{0}_3 & \mathbb{1}_3 \\ \mathbb{1}_3 & \mathbb{0}_3 \end{pmatrix},\ A_3 = \begin{pmatrix} Y & Y \\ Y & Y \end{pmatrix},\ \text{where } Y = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \qquad (5)$$

The irreducible decomposition of the representation is $\underline{\mathbf{6}} = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3}$. The complete set of primitive orthogonal idempotents can be written in the basis (5) as follows

$$B_{\mathbf{1}} = \frac{1}{6}\,(A_1 + A_2 + A_3)\,, B_{\mathbf{2}} = \frac{1}{3}\left(A_1 + A_2 - \frac{1}{2}A_3\right), B_{\mathbf{3}} = \frac{1}{2}\,(A_1 - A_2)\,. \qquad (6)$$

Program Output. The calculation was performed on PC with 3.30GHz CPU and 16GB RAM. The superscripts in the list of '`Irreducible dimensions`' represent the numbers of equal dimensions. The expressions for primitive idempotents are tensor product polynomials in the matrices (6). '`Checksum`' is the sum of all dimensions, which should coincide with the dimension of the representation.

Wreath product $\mathsf{S}_4(\textit{octahedron}) \wr \mathsf{A}_5(\textit{icosahedron})$
`Representation dimension:` **2 176 782 336**
`Rank:` **122 776**
`Wreath product decomposition is multiplicity free`
`Number of irreducible components:` 122 776
`Number of different dimensions:` 134
`Irreducible dimensions:`

$1, 4^{6}, 6^{3}, 8^{6}, 9^{3}, 12^{15}, 16^{32}, 18^{7}, 20, 24^{70}, 32^{41}, 36^{86}, 45, 48^{191}, 54^{26}, 64^{84}, 72^{298}, 80^{4}, 81^{7}, 96^{412}, 108^{223}, 128^{114}, 144^{913}, 162^{54}, 180^{8}, 192^{704}, 216^{926}, 243^{4}, 256^{104}, 288^{1804}, 320^{7}, 324^{504}, 384^{772}, 405^{4}, 432^{2517}, 486^{99}, 512^{76}, 576^{2508}, 648^{1909}, 720^{17}, 729^{9}, 768^{705}, 864^{4303}, 972^{818}, 1024^{51}, 1152^{2562}, 1280^{3}, 1296^{4455}, 1458^{141}, 1536^{479}, 1620^{16}, 1728^{5322}, 1944^{2712}, 2048^{20}, 2187^{4}, 2304^{1935}, 2592^{6708}, 2880^{14}, 2916^{961}, 3072^{223}, 3456^{4575}, 3645^{7}, 3888^{5495}, 4096^{4}, 4374^{136}, 4608^{1004}, 5120, 5184^{6924}, 5832^{2754}, 6144^{59}, 6480^{18}, 6561^{9}, 6912^{2719}, 7776^{\underline{\mathbf{6966}}}, 8192^{3}, 8748^{822}, 9216^{329}, 10368^{4760}, 11520^{10}, 11664^{4695}, 12288^{19}, 13122^{98}, 13824^{1011}, 14580^{13}, 15552^{5781}, 17496^{1999}, 18432^{83}, 19683^{3}, 20736^{2085}, 23328^{4826}, 25920^{16}, 26244^{511}, 27648^{260}, 31104^{2964},$

$32805^{3}, 34992^{2775}, 36864^{5}, 39366^{55}, 41472^{534}, 46080, 46656^{3012}, 52488^{1023}, 55296^{15}, 58320^{19}, 59049^{5}, 62208^{877}, 69984^{2173}, 78732^{242}, 82944^{48}, 93312^{1038}, 103680^{4}, 104976^{1079}, 118098^{27}, 124416^{102}, 131220^{8}, 139968^{905}, 157464^{355}, 186624^{130}, 209952^{568}, 233280^{7}, 236196^{84}, 279936^{148}, 295245, 314928^{254}, 354294^{6}, 419904^{116}, 472392^{79}, 524880^{3}, 531441, 629856^{62}, 708588^{15}, 944784^{26}, 1180980, \underline{\mathbf{1417176}}^{9}$

`Checksum =` 2176782336 `Maximum number of equal dimensions =` 6966

`Wreath irreducible projectors:`

$$
\begin{aligned}
\widetilde{B}_{1} =& B_1^{\otimes 12}\\
\widetilde{B}_{2} =& B_1^{\otimes 3}\otimes B_2\otimes B_1^{\otimes 6}\otimes B_2\otimes B_1\\
\widetilde{B}_{3} =& B_1^{\otimes 9}\otimes B_2\otimes B_1^{\otimes 2}+B_1^{\otimes 4}\otimes B_2\otimes B_1^{\otimes 7}\\
&\vdots\\
\widetilde{B}_{61387} =& B_2\otimes B_3\otimes B_1\otimes B_2^{\otimes 2}\otimes B_1^{\otimes 3}\otimes B_3^{\otimes 2}\otimes B_2\otimes B_3\\
&+B_3\otimes B_2\otimes B_1\otimes B_2\otimes B_1\otimes B_3\otimes B_1\otimes B_2\otimes B_3\otimes B_1\otimes B_3\otimes B_2\\
&+B_1^{\otimes 2}\otimes B_3\otimes B_2\otimes B_3^{\otimes 2}\otimes B_2\otimes B_3\otimes B_1\otimes B_2^{\otimes 2}\otimes B_1\\
&+B_1\otimes B_2^{\otimes 2}\otimes B_3\otimes B_1\otimes B_3^{\otimes 2}\otimes B_2\otimes B_3\otimes B_1\otimes B_2\otimes B_1\\
\widetilde{B}_{61388} =& B_2^{\otimes 2}\otimes B_3\otimes B_1^{\otimes 2}\otimes B_3\otimes B_1\otimes B_2\otimes B_3\otimes B_2^{\otimes 2}\otimes B_3\\
&+B_1\otimes B_2\otimes B_3\otimes B_2^{\otimes 2}\otimes B_3\otimes B_2^{\otimes 2}\otimes B_3\otimes B_1^{\otimes 2}\otimes B_3\\
\widetilde{B}_{61389} =& B_1^{\otimes 2}\otimes B_2^{\otimes 3}\otimes B_1\otimes B_2\otimes B_3\otimes B_2\otimes B_1\otimes B_3^{\otimes 2}\\
&+B_2\otimes B_1\otimes B_2\otimes B_3\otimes B_2\otimes B_1\otimes B_3\otimes B_2\otimes B_1^{\otimes 2}\otimes B_2\otimes B_3\\
&+B_1\otimes B_3\otimes B_2\otimes B_1^{\otimes 3}\otimes B_3\otimes B_2^{\otimes 3}\otimes B_3\otimes B_2\\
&+B_3\otimes B_2\otimes B_3\otimes B_2\otimes B_1^{\otimes 2}\otimes B_2\otimes B_1^{\otimes 2}\otimes B_2\otimes B_3\otimes B_2\\
&+B_2\otimes B_3^{\otimes 3}\otimes B_1\otimes B_2\otimes B_1^{\otimes 3}\otimes B_2^{\otimes 3}\\
&+B_3\otimes B_2^{\otimes 2}\otimes B_3\otimes B_2^{\otimes 2}\otimes B_1\otimes B_3\otimes B_1^{\otimes 3}\otimes B_2\\
&\vdots\\
\widetilde{B}_{122774} =& B_3^{\otimes 2}\otimes B_2\otimes B_3^{\otimes 9}+B_3^{\otimes 3}\otimes B_2\otimes B_3^{\otimes 8}+B_3^{\otimes 10}\otimes B_2\otimes B_3+B_3^{\otimes 11}\otimes B_2\\
\widetilde{B}_{122775} =& B_3^{\otimes 2}\otimes B_2\otimes B_3\otimes B_2\otimes B_3^{\otimes 7}+B_3^{\otimes 2}\otimes B_2\otimes B_3^{\otimes 2}\otimes B_2\otimes B_3^{\otimes 6}\\
&+B_3^{\otimes 3}\otimes B_2\otimes B_3^{\otimes 4}\otimes B_2\otimes B_3^{\otimes 3}+B_3^{\otimes 5}\otimes B_2\otimes B_3^{\otimes 4}\otimes B_2\otimes B_3\\
&+B_3^{\otimes 8}\otimes B_2\otimes B_3^{\otimes 2}\otimes B_2+B_3^{\otimes 9}\otimes B_2\otimes B_3\otimes B_2\\
\widetilde{B}_{122776} =& B_3^{\otimes 3}\otimes B_2^{\otimes 2}\otimes B_3^{\otimes 7}+B_2\otimes B_3^{\otimes 4}\otimes B_2\otimes B_3^{\otimes 6}\\
&+B_3\otimes B_2\otimes B_3^{\otimes 3}\otimes B_2\otimes B_3^{\otimes 6}+B_3^{\otimes 6}\otimes B_2\otimes B_3\otimes B_2\otimes B_3^{\otimes 3}\\
&+B_3^{\otimes 7}\otimes B_2^{\otimes 2}\otimes B_3^{\otimes 3}+B_3^{\otimes 9}\otimes B_2^{\otimes 2}\otimes B_3
\end{aligned}
$$

`Time: 0.58 sec`

`Maximum number of tensor monomials: 531441`

For comparison, we give a part of the output for a somewhat larger problem.

Wreath product $\mathsf{A}_5(icosahedron) \wr \mathsf{A}_5(icosahedron)$
`Representation dimension:` **8 916 100 448 256**
`Rank:` **3 875 157**
`Wreath product decomposition is multiplicity free`
`Number of irreducible components:` 3 875 157
`Number of different dimensions:` 261
`Time: 7.35 sec`
`Maximum number of tensor monomials: 16777216`

5. Conclusion

One of the main goals of the work was to develop a tool for the study of models of multipartite quantum systems. The projection operators obtained by the program are matrices of huge dimension. Obviously, the explicit calculation of such matrices is impossible. However, the expression of projectors for wreath products in the form of tensor polynomials makes it possible to reduce the computation of quantum correlations to a sequence of computations with small matrices of local projectors.

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Cluster monomials and Schur positivity

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1 Schur functions and Schur positivity

Schur polynomials $s_\lambda(x_1, \dots, x_N)$ form a vector space basis (over $\mathbb{Z}$) of the ring o metric polynomials in the variables $x_1, \dots x_N$, while λ runs over the set of partiti Young diagrams.

A symmetric function is *Schur positive* if its expansion on the basis of the functions involves only non-negative coefficients only.

Since a product of symmetric polynomials is symmetric, one can expand it in of Schur polynomials. In particular, for product of Schur functions, we get

$$s_\mu(x_1, \dots x_N)s_\nu(x_1, \dots x_N) = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda(x_1, \dots x_N).$$

Here $c^\lambda_{\mu,\nu}$ are the Littlewood-Richardson coefficients. These coefficients are nonne since we have a combinatorial description of these coefficients : $c^\lambda_{\mu,\nu}$ is equal number of semistandard Young tableaux of skew shape $\lambda \setminus \mu$ and of weight ν.

For a quadruple of partitions μ, ν, μ', ν', we have

$$s_\mu s_\nu - s_{\mu'} s_{\nu'} = \sum_\lambda (c^\lambda_{\mu,\nu} - c^\lambda_{\mu',\nu'}) s_\lambda(x_1, \dots x_N).$$

Then the LHS is said to be *Schur positive* if for any λ,

$$c^\lambda_{\mu,\nu} - c^\lambda_{\mu',\nu'} \geq 0.$$

Questions on Schur positivity of several types of expressions $s_\mu s_\nu - s_{\mu'} s_{\nu'}$ has been and studied in series of works, see [2, 5, 3, 4].

Lam, Postnikov, and Pyaljavskii [4] proved Schur positivity of

$$s_{\mu\vee\nu} s_{\mu\wedge\nu} - s_\mu s_\nu,$$

and confirmed several open problems posed in [2, 5, 3, 4].

Recall that a flag minor is a minor of a matrix which is equal to a determina square submatrix which is constituted of elements of the intersection of a set of co and the first consecutive rows. To a flag minor Δ_I, where $I = \{i_1, i_2, \dots, i_k\}$ denot column set, is associated a Schur function s_λ with $\lambda = (i_k - k + 1, i_{k-1} - (k-1) + 1,$

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Definition 1.1 *A polynomial in flag minors is* Schur positive *if substituting Schur functions instead of the flag minors yields a Schur positive symmetric fu*

For example, for a subsets $I = \{i_1, i_2, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$ and $\{\max(i_1, j_1), \ldots, \max(i_k, j_k)\}$, $I \wedge J = \{\min(i_1, j_1), \ldots, \min(i_k, j_k)\}$, we have Sc tivity of the polynomial

$$\Delta_{I\vee J}\Delta_{I\wedge J} - \Delta_I\Delta_J.$$

Namely, this follows from Schur positivity of (3).

2 Cluster algebra of $\mathbb{C}[SL_N^{w_0,e}]$

Let B be the subgroup of upper triangular matrices of SL_N and B_- be that triangular, w_0 be the longest element of the Weyl group, the group of permut $[N]$. The coordinate ring the big double Bruhat cell $SL_N^{w_0,e} := B \cap B_- w_0 B_-$, $\mathbb{C}$ has a cluster algebra structure [1].

The cluster variables of such a cluster algebra are specific polynomials in fla We call *cluster polynomials* such polynomials.

We state the following conjectures bases on numerous computer experimen

Conjecture 2.1 *Any cluster polynomial is Schur positive.*

Conjecture 2.2 *Any such a polynomial as a linear combination of Schur fun the full support. Namely, for a cluster polynomial, all integer points of the conv vectors corresponding to partitions which support the Schur functions of the corr linear combination correspond to summands with positive coefficients.*

3 Some examples

We choose an initial cluster seed being the seed corresponding to the reduced de tion $s_1 s_2 s_1 \cdots s_{N-1} \cdots s_1$ of w_0 (see [1]). Any sequence of cluster mutations at vertices results in a flag minor. Because of that, we get Schur positivity in suc

A mutation at a 6-valency vertex of the initial seed gives a cluster polynom form

$$\Delta_{i-1,[i+1,i+j+1]}\Delta_{[i,i+j-1]} - \Delta_{i-1,[i+1,i+j-1]}\Delta_{[i,i+j+1]}.$$

Substituting Schur functions instead of the flag minors yields the following pc in Schur functions

$$s_{(i^{j+1},i-1)}s_{i^j} - s_{i^{j+2}}s_{(i^{j-1},i-1)},$$

where a^b denotes sequence $(a, a, \ldots, a)$ with b entries.

For example, for $i = 3$, $j = 2$, we get Schur positivity of (4) as well as v Conjecture 2.2.

Schur positivity of (4) can be obtained using a result of ([6]), for which foll

$$\Delta_I \Delta_J - \Delta_{I'} \Delta_{J'}$$

is Schur positive if, for any interval $K \subset \mathbb{N}$, there holds

$$\max(|I \cap K|, |J \cap K|) \leq \max(|I' \cap K|, |J' \cap K|).$$

Observation. All cluster variables of the form (5) computed in our exp fulfill (6), and thus are Schur positive.

4 Extreme Schur functions of cluster polynom

For partitions ν and μ, the Schur functions of the product $s_\nu s_\mu$ labeled by the s and the concatenation $\mu \cup \nu$ are lexmax and lexmin terms, respectively.

4.1 Two tropical semirings on partitions or Young diag

Lets us defined two dual tropical semirings on the set of partitions: for one we the pair of operations as summation and lexmax, and for another as concation lexmin.

Namely the first tropical semiring is specified by

$$(\lambda_1, \ldots, \lambda_k) \odot_1 (\mu_1, \ldots, \mu_k) = (\lambda_1, \ldots, \lambda_k) + (\mu_1, \ldots, \mu_k) = (\lambda_1 + \mu_1, \ldots, \lambda_k$$

and

$$(\lambda_1, \ldots, \lambda_k) \oplus_1 (\mu_1, \ldots, \mu_k) = \max\{(\lambda_1, \ldots, \lambda_k), (\mu_1, \ldots, \mu_k)\},$$

where max is the lexicographical maximum of two vectors of $\mathbb{R}^k$.

The second semiring is specified by

$$(\lambda_1, \ldots, \lambda_k) \odot_2 (\mu_1, \ldots, \mu_k) = (\lambda_1, \ldots, \lambda_k) \cup (\mu_1, \ldots, \mu_k) = (\lambda_1, \ldots, \lambda_k, \mu_1, . .$$

where $(x_1, \ldots, x_n)^\uparrow = (x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)})$ is a non-increasing ordering of x_i's.

$$(\lambda_1, \ldots, \lambda_k) \oplus_2 (\mu_1, \ldots, \mu_k) = \min\{(\lambda_1, \ldots, \lambda_k), (\mu_1, \ldots, \mu_k)\},$$

where min is the lexicographical minimum of two vectors of $\mathbb{R}^k$.

These semirings are dual with respect to transposition. Namely, for a $\lambda = (\lambda_1, \ldots, \lambda_k)$, let λ' denote the transposed (dual) partition, $\lambda'_i := \#I$, wher such that $\lambda_j \geq i$, for $j \in I$. Then

$$\lambda \odot_2 \nu = (\lambda' \odot_1 \nu')', \quad \lambda \oplus_2 \nu = (\lambda' \oplus_1 \nu')'.$$

4.2

Theorem 4.1 *Let Q^{in} be the initial seed of the cluster algebra $\mathbb{C}[SL_N^{w_0,e}]$.Le cluster polynomial in flag minors. Then the expansion of the Schur specializa on the basis of Schur functions has terms labeled by lexmin and lexmax partit coefficients* 1*. Moreover these lexmin and lexmax partitions can be obtained by the same sequence of mutations from Q^{in} as for a cluster variables correspon but with respect to the two tropical semirings on the partitions.*

For example,

$$\frac{\left(s_{(5,3,2,2,2)} + s_{(6,3,2,2,1)} + s_{(7,3,2,2)}\right) s_{(4,4,3)}s_{(3,3)} + s_{(3)}s_{(3,3,3,3)}s_{(2,2,2)}s_{(5,5)}}{\begin{array}{c} s_{(4,3,3,2,2,2)}+s_{(4,4,2,2,2,2)}+s_{(4,4,3,2,2,1)}+s_{(5,3,3,2,2,1)}+s_{(5,4,2,2,2,1)}+s_{(5,4,3,2,1,1)} \\ +s_{(5,4,3,2,2)}+s_{(5,5,2,2,1,1)}+s_{(5,5,3,2,1)}+s_{(6,3,3,2,2)}+s_{(6,4,2,2,2)} \\ +s_{(6,4,3,2,1)}+s_{(6,5,2,2,1)}+s_{(6,5,3,2)}+s_{(6,6,2,2)} \end{array}}$$
$$= s_{(5,4,3,3)} + s_{(6,4,3,2)} + s_{(7,4,3,1)} + s_{(8,4,3)}$$

the lexmin of concatenations of the numerator is $(5, 4, 4, 3, 3, 3, 3, 2, 2, 2)$, and su lexmin of denominator $(4, 3, 3, 2, 2, 2)$ yields $(5, 4, 3, 3)$ of RHS; lexmax sums, t nator yields $(14, 10, 5, 2)$, the denominator $(6, 6, 2, 2)$, and we get $(8, 4, 3)$ of R that the support here is the segment $[(5, 4, 3, 3), (8, 4, 3)]$.

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Using tropical optimization in rank-one approximation of non-negative matrices

N. Krivulin and E. Romanova

Introduction

Low rank approximation of matrices finds wide use in many areas [1], such as machine learning, statistics, and data compression. In many applications, approximation by matrices of unit rank is of interest to deal only with the basic information involved in the data under consideration [2, 3].

In [4, 5], the problem of rank-one approximation of positive square matrices is formulated as a problem of minimizing the log-Chebychev distance between matrices. The optimization problem is represented in terms of tropical (idempotent) mathematics, which deals with the theory and applications of idempotent semifields. A solution approach based on methods and results of tropical optimization is used to provide a complete direct solution given in compact vector form.

In this paper, we extend the above results to solve the rank-one approximation problem in the case of rectangular non-negative matrices. The problem is formulated in terms of max-algebra, which is a tropical semifield with the maximum in the role of addition, and with multiplication defined as usual. We start with necessary definitions and results of tropical mathematics, and then apply them to obtain complete solutions to a tropical optimization problem under different assumptions. The results obtained serve as the basis to derive a solution to the approximation problem in question in compact closed vector form. We offer the solution in different forms for the case of arbitrary non-negative matrices and for the case of matrices without zero columns.

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1. Tropical algebra

We begin with preliminary definitions and results of tropical mathematics from [4, 6], which are used in what follows.

1.1. Idempotent semifield

Suppose $\mathbb{X}$ is a nonempty set that is closed under addition $\oplus$ and multiplication $\otimes$. Addition and multiplication are associative and commutative, and have respective neutral elements, zero $\mathbb{0}$ and identity $\mathbb{1}$. Addition is idempotent, resulting in the equality $x \oplus x = x$ for all $x \in \mathbb{X}$. Multiplication distributes over addition, and is invertible in the sense that, for each nonzero element $x \in \mathbb{X}$, there exists inverse element x^{-1} such that $x \otimes x^{-1} = \mathbb{1}$. Together with the operations $\oplus$ and $\otimes$, and their neutral elements, the set $\mathbb{X}$ forms the algebraic system, which is usually called the idempotent semifield. In what follows, the multiplication sign $\otimes$ is dropped for simplicity.

The semifield $\mathbb{R}_{\max,\times}$ is defined on the set of non-negative real numbers, and equipped with the addition $\oplus$ defined as maximum, and the multiplication $\otimes$ defined as usual. The neutral elements $\mathbb{0}$ and $\mathbb{1}$ coincide with the arithmetic zero 0 and one 1. The power and inversion notations have the usual meaning. This semifield is often called max-algebra.

1.2. Matrix algebra

Let $\mathbb{X}^{m \times n}$ be the set of matrices over $\mathbb{X}$, with m rows and n columns. A matrix with all zero elements is the zero matrix denoted $\mathbf{0}$. A matrix with $\mathbb{1}$ on the diagonal and $\mathbb{0}$ elsewhere is identity matrix, which is denoted by $\boldsymbol{I}$. In the case of max-algebra, the zero and identity matrices have the usual form. Any matrix without zero columns is called column-regular.

Matrix addition and multiplication, and multiplication by scalars are defined as usual, except that the arithmetic operations are replaced by $\oplus$ and $\otimes$.

The multiplicative conjugate transpose of a nonzero matrix $\boldsymbol{A} = (a_{ij}) \in \mathbb{X}^{m \times n}$ is the matrix $\boldsymbol{A}^{-} = (a_{ij}^{-}) \in \mathbb{X}^{n \times m}$ with the elements $a_{ij}^{-} = a_{ji}^{-1}$ if $a_{ji} \neq \mathbb{0}$, and $a_{ij}^{-} = \mathbb{0}$ otherwise.

Consider a square matrix $\boldsymbol{A} = (a_{ij}) \in \mathbb{X}^{n \times n}$. The trace of the matrix $\boldsymbol{A}$ is calculated as $\operatorname{tr} \boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn}$.

The spectral radius of $\boldsymbol{A}$ is the scalar $\lambda = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr}^{1/n}(\boldsymbol{A}^n)$.

The Kleene star operator for $\boldsymbol{A}$ is given by the formula $\boldsymbol{A}^{*} = \boldsymbol{I} \oplus \cdots \oplus \boldsymbol{A}^{n-1}$.

The set of column vectors of order n is denoted by $\mathbb{X}^n$. A vector that has no zero elements is called regular. In max-algebra, the regularity of a vector means that the vector is positive.

The multiplicative conjugate transpose of a nonzero column vector $\boldsymbol{x} = (x_i)$ is a row vector $\boldsymbol{x}^{-} = (x_i^{-})$, where $x_i^{-} = x_i^{-1}$ if $x_i \neq \mathbb{0}$, and $x_i^{-} = \mathbb{0}$ otherwise.

2. Tropical optimization problem

Given a rectangular matrix $\boldsymbol{A} \in \mathbb{X}^{m\times n}$, the problem is to find regular vectors $\boldsymbol{x} \in \mathbb{X}^m$ and $\boldsymbol{y} \in \mathbb{X}^n$ that achieve the minimum

$$\min_{\boldsymbol{x},\boldsymbol{y}} \quad \boldsymbol{x}^- \boldsymbol{A}\boldsymbol{y} \oplus \boldsymbol{y}^- \boldsymbol{A}^- \boldsymbol{x}. \tag{1}$$

The following theorem generalizes the result of the paper [5], and gives a complete solution to problem (1) in explicit form.

Theorem 1. *Let $\boldsymbol{A} \in \mathbb{X}^{m\times n}$ be a nonzero matrix, μ be the spectral radius of the matrix $\boldsymbol{A}\boldsymbol{A}^-$. Then, the minimum in problem* (1) *is equal to $\mu^{1/2}$, and all regular solutions are given by*

$$\boldsymbol{x} = (\mu^{-1}\boldsymbol{A}\boldsymbol{A}^-)^*\boldsymbol{v} \oplus \mu^{-1/2}\boldsymbol{A}(\mu^{-1}\boldsymbol{A}^-\boldsymbol{A})^*\boldsymbol{w},$$
$$\boldsymbol{y} = \mu^{-1/2}\boldsymbol{A}^-(\mu^{-1}\boldsymbol{A}\boldsymbol{A}^-)^*\boldsymbol{v} \oplus (\mu^{-1}\boldsymbol{A}^-\boldsymbol{A})^*\boldsymbol{w}; \qquad \boldsymbol{v} \in \mathbb{X}^m, \quad \boldsymbol{w} \in \mathbb{X}^n.$$

For column-regular matrices, the solution can be represented as follows.

Theorem 2. *Let $\boldsymbol{A} \in \mathbb{X}^{m\times n}$ be a column-regular matrix, μ be the spectral radius of $\boldsymbol{A}\boldsymbol{A}^-$. Then, the minimum in problem* (1) *is equal to $\mu^{1/2}$, and all regular solutions are given by*

$$\boldsymbol{x} = (\mu^{-1}\boldsymbol{A}\boldsymbol{A}^-)^*\boldsymbol{u}, \quad \boldsymbol{u} \in \mathbb{X}^m;$$
$$\mu^{-1/2}\boldsymbol{A}^-\boldsymbol{x} \le \boldsymbol{y} \le \mu^{1/2}(\boldsymbol{x}^-\boldsymbol{A})^-.$$

3. Application to matrix approximation

The problem of approximating a non-negative rectangular matrix $\boldsymbol{A} = (a_{ij})$ by positive matrix $\boldsymbol{X} = (x_{ij})$ is formulated to minimize the Chebyshev distance in logarithmic scale, given by

$$\max_{i,j:a_{ij}\neq 0} |\log a_{ij} - \log x_{ij}| = \log \max_{i,j:a_{ij}\neq 0} \max(a_{ij}x_{ij}^{-1}, a_{ij}^{-1}x_{ij}).$$

Since the logarithm (on the base greater than one) is monotone increasing, the approximation problem is equivalent to minimizing the argument of the logarithm. Observing that any positive matrix $\boldsymbol{X}$ of unit rank can be represented as $\boldsymbol{s}\boldsymbol{t}^T$, where $\boldsymbol{s} = (s_i)$ and $\boldsymbol{t} = (t_j)$ are positive vectors, we reduce the problem to that of the form

$$\min_{\boldsymbol{s},\boldsymbol{t}} \quad \max_{i,j:a_{ij}\neq 0} \max(s_i^{-1}a_{ij}t_j^{-1}, s_i a_{ij}^{-1}t_j).$$

Representation of the objective function in terms of max-algebra yields

$$\bigoplus_{i,j:a_{ij}\neq 0} (s_i^{-1}a_{ij}t_j^{-1} \oplus s_i a_{ij}^{-1}t_j) = \boldsymbol{s}^-\boldsymbol{A}(\boldsymbol{t}^-)^T \oplus \boldsymbol{t}^T\boldsymbol{A}^-\boldsymbol{s}.$$

We now formulate the rank-one approximation problem as to find a matrix $\boldsymbol{X} = \boldsymbol{s}\boldsymbol{t}^T$, where $\boldsymbol{s}$ and $\boldsymbol{t}$ are positive vectors that solve the problem

$$\min_{\boldsymbol{s},\boldsymbol{t}} \boldsymbol{s}^-\boldsymbol{A}(\boldsymbol{t}^T)^- \oplus \boldsymbol{t}^T\boldsymbol{A}^-\boldsymbol{s}.$$

The last problem has the same form as problem (1) with $\boldsymbol{x} = \boldsymbol{s}$, $\boldsymbol{y} = (\boldsymbol{t}^T)^-$ and thus admits complete solutions given by the results of section 2.

Theorem 3. *Let $\boldsymbol{A}$ be a non-negative matrix, μ be the spectral radius of the matrix $\boldsymbol{A}\boldsymbol{A}^-$. Then, the minimum error of* log*-Chebyshev approximation is equal to* $\log(\mu)/2$*, and all approximate matrices are given by $\boldsymbol{s}\boldsymbol{t}^T$, where*

$$\boldsymbol{s} = (\mu^{-1}\boldsymbol{A}\boldsymbol{A}^-)^*\boldsymbol{v} \oplus \mu^{-1/2}\boldsymbol{A}(\mu^{-1}\boldsymbol{A}^-\boldsymbol{A})^*\boldsymbol{w},$$

$$\boldsymbol{t}^T = (\mu^{-1/2}\boldsymbol{A}^-(\mu^{-1}\boldsymbol{A}\boldsymbol{A}^-)^*\boldsymbol{v} \oplus (\mu^{-1}\boldsymbol{A}^-\boldsymbol{A})^*\boldsymbol{w})^-; \qquad \boldsymbol{v} \in \mathbb{X}^m, \quad \boldsymbol{w} \in \mathbb{X}^n.$$

The next result holds if the approximated matrix has no zero columns.

Theorem 4. *Let $\boldsymbol{A}$ be a non-negative matrix without zero columns, μ be the spectral radius of $\boldsymbol{A}\boldsymbol{A}^-$. Then, the minimal error of* log*-Chebyshev approximation is equal to* $\log \mu^{1/2}$*, and all approximate matrices are given by $\boldsymbol{s}\boldsymbol{t}^T$, where*

$$\boldsymbol{s} = (\mu^{-1}\boldsymbol{A}\boldsymbol{A}^-)^*\boldsymbol{u}, \quad \boldsymbol{u} \in \mathbb{X}^m;$$

$$\mu^{-1/2}\boldsymbol{s}^-\boldsymbol{A} \le \boldsymbol{t}^T \le \mu^{1/2}(\boldsymbol{A}^-\boldsymbol{s})^-.$$

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Quick triangular orthogonal decomposition of matrices

Gennadi Malaschonok and Gurgen Gevondov

Abstract. A new algorithm for calculating the triangular orthogonal decomposition of matrices is proposed. It differs from previously known algorithms by the smallest asymptotic complexity.

Introduction

The problem of the orthogonal decomposition of matrices is still known as the QR-decomposition problem. It is one of the subtasks that are associated with spectral decomposition. Given the matrix A, it is required to represent it as a product of two factors, A=QR, where Q is a unitary matrix (orthogonal in the case of real numbers), R is an upper triangular matrix. The algorithm of the QR-decomposition should not be confused with the QR-algorithm, that is the algorithm for calculating the spectrum of the matrix (singular value decomposition). There are a large number of different approaches [1]-[4] to the problem of computing the orthogonal decomposition, including fast recursive algorithm [5]. However, the best-known algorithms in terms of the number of operations are algorithms that have cubic complexity. In this paper, we consider an algorithm of orthogonal decomposition, which has the complexity of matrix multiplication.

Let A be a matrix over a field. It is required to find the upper triangular matrix R and the orthogonal (unitary if the initial field is a field of complex numbers) Q matrix such that $A = QR$.

For definiteness, we will consider an algorithm applied to a square matrix A over a field of real numbers.

Consider the case of a 2×2 matrix. The desired decomposition $A = QR$ has the form:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where the numbers s and c satisfy the equation $s^2 + c^2 = 1$.

After multiplying from the left of both sides of the equation by the inverse matrix $Q^{-1} = Q^T$, we get: $Q^T A = R$.

If $\gamma = 0$ then we can set $c = 1,\ s = 0$. If $\gamma \neq 0$, then $\Delta = \alpha^2 + \gamma^2 > 0$. Then we get $c\alpha + s\gamma = a,\quad c\gamma - s\alpha = 0$ and $c = a\alpha/\Delta,\quad s = a\gamma/\Delta$.

Therefore, $1 = s^2 + c^2 = a^2/\Delta$, hence $|a| = \sqrt{\Delta}$. $c = \alpha/\sqrt{\Delta},\quad s = \gamma/\sqrt{\Delta}$.

We denote such a matrix Q by $g_{\alpha,\gamma}$.

1. Sequential QR decomposition

Let the matrix A be given, its elements (i, j) and $(i+1, j)$ be α and γ, and all the elements to the left of them be zero: $\forall(s < j) : (a_{i,s} = 0)\ \&\ (a_{i+1,s} = 0)$.

Let $G_{i,j} = \mathbf{diag}(I_{i-1}, g_{\alpha,\gamma}, I_{n-i-1})$. Then the matrix $G_{i,j}A$ differs from A only in two rows i and $i+1$, but all the elements to the left of the column j remain zero, and in the column j in $i+1$ line will be 0.

This property of the Givens matrix allows us to formulate such an algorithm

Algorithm

(1). First we reset the elements under the diagonal in the left column:

$$A_1 = G_{1,1}G_{2,1}...G_{n-2,1}G_{n-1,1}A$$

(2). Then we reset the elements that are under the diagonal in the second column:

$$A_2 = G_{2,2}G_{3,2}...G_{n-2,2}G_{n-1,2}A_1$$

(k). Denote $G_{(k)} = G_{k,k}G_{k-1,k}...G_{n-2,k}G_{n-1,k},\ \ k = 1, 2, .., n-1$. Then, to calculate the elements of the k th column, we need to obtain the product of matrices

$$A_k = G_{(k)}A_{k-1}.$$

(n-1). At the end of the calculation, the element in the $n-1$ column will be reseted: $A_{n-1} = G_{(n-1)}A_{n-2} = G_{n-1,n-1}A_{n-2}$.

You can find the number of operations. It is necessary to calculate the $(n^2 - n)/2$ turn matrices and for each of them 6 operations must be performed. when calculating A_1, the number of multiplications of the Givens matrices into columns of two elements (4 multiplications and 2 additions) is $(n-1)^2$. When calculating A_2, the number of such multiplications is $(n-2)^2$, and so on. As a result, we get

$$6(n^2 - n)/2 + 6\sum_{i=1..n-1} i^2 = 3n^2 - 3n + 6(n-1)(2n-1)n/6 \approx 2n^3$$

Here we count the number of all arithmetic operations and the operations of extracting the square root.

2. QR_G decomposition

Let a matrix M of size $2n \times 2n$ be divided into four equal blocks: $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.
There are three steps in this algorithm.

Algorithm

(1). The first stage is the QR_G decomposition of the block C:

$$C = Q_1 C_1,\ M_1 = \mathbf{diag}(I, Q_1)M = \begin{pmatrix} A & B \\ C_1 & D_1 \end{pmatrix}.$$

(2). The second stage is the cancellation of a parallelogram composed of two triangular blocks: the lower triangular part A^L of the block A and the upper triangular part C_1^U of the block C_1. Denote the upper triangular matrix A_1 and annihilating matrix Q_2:

$$\begin{pmatrix} A \\ C_1 \end{pmatrix} = \begin{pmatrix} A_1 \\ 0 \end{pmatrix},\ M_2 = Q_2 M_1 = \begin{pmatrix} A_1 & B_1 \\ 0 & D_2 \end{pmatrix}.$$

(3). The third stage is the QR_G decomposition of the D_2 block: $D_2 = Q_3 D_3$.

$$R = \mathbf{diag}(I, Q_3)M_2 = \begin{pmatrix} A_1 & B_1 \\ 0 & D_3 \end{pmatrix}.$$

As a result, we get:

$$M = Q^T R,\quad Q = \mathbf{diag}(I, Q_3)Q_2\,\mathbf{diag}(I, Q_1).$$

Since the first and third stages are recursive calls of the QR_G -procedures, it remains to describe the parallelogram cancellation procedure. Let's call it a QP decomposition.

3. QP-decomposition

Let the matrix $M = \begin{pmatrix} A \\ B^U \end{pmatrix}$ have dimensions $2n \times n$ and, at the same time, the lower unit B^U of size $n \times n$, n -countable, has an upper triangular shape - all elements under its main diagonal are zero. We are looking for the factorization of the matrix $M = QP = Q\begin{pmatrix} A^U \\ 0 \end{pmatrix}$, with the orthogonal matrix Q.

It is required to annul all elements between the upper and lower diagonals of the M matrix, including the lower diagonal. It is easy to see that this can be done with Givens matrices. We will consistently perform column invalidation by traversing column elements from bottom to top and traversing columns from left to right.

But we are interested in the block procedure. Since n is even, we can break the parallelogram formed by the diagonals into 4 parts using its two middle lines. We get 4 equal parallelograms. To cancel each of them, we will simply call the

parallelogram cancellation procedure 4 times. We will perform the calculations in this order: the bottom left (P_{ld}), then we simultaneously cancel the top left (P_{lu}) and the bottom right (P_{rd}), and last we will cancel the top right (P_{en}). The corresponding orthogonal Givens matrices of size $n \times n$ are denoted Q_{ld}. Q_{lu}. Q_{rd} and Q_{ru}. Let

$$\bar{Q}_{ld} = \mathbf{diag}(I_{n/2}, Q_{ld}, I_{n/2}), \quad \bar{Q}_{ru} = \mathbf{diag}(I_{n/2}, Q_{ru}, I_{n/2}),$$

As a result, we get:

$$Q = \bar{Q}_{ru}\, \mathbf{diag}(Q_{lu}, Q_{rd})\bar{Q}_{ld}$$

The number of multiplications of matrix blocks of size $n/2 \times n/2$ is 24. Hence the total number of operations: $Cp(2n) = 4Cp(n) + 24M(n/2)$.. Suppose that for multiplication of two matrices of size $n \times n$ you need γn^β operations and $n = 2^k$, then we get: $Cp(2^{k+1}) = 4Cp(2^k) + 24M(2^{k-1}) = 4^k Cp(2^1) + 24\gamma \sum_{i=0}^{k-1} 4^{k-i-1}2^{i\beta} = 24\gamma(n^2/4)\frac{2^{k(\beta-2)}-1}{2^{(\beta-2)}-1} + 6n^2 = 6\gamma\frac{n^\beta - n^2}{2^\beta - 4} + 6n^2$

$$Cp(n) = \frac{6\gamma n^\beta}{2^\beta(2^\beta - 4)} + \frac{3n^2}{2}(1 - \frac{\gamma}{2^\beta - 4})$$

4. The complexity of QR_G decomposition

Let us estimate the number of operations $C(n)$ in this block-recursive decomposition algorithm, assuming that the complexity of the matrix multiplication is $M(n) = \gamma n^\beta$, the complexity of canceling the parallelogram is $Cp(n) = \alpha n^\beta$, where α, β, γ are constants, $\alpha = \frac{6\gamma}{2^\beta(2^\beta - 4)}$ and $n = 2^k$:

$$C(n) = 2C(n/2) + Cp(n) + 6M(n/2) = 2C(2^{k-1}) + Cp(2^k) + 6M(2^{k-1}) =$$

$$C(2^0)2^k + \sum_{i=0}^{k} 2^{k-i}Cp(2^i) + 6\sum_{i=0}^{k} 2^{k-i}M(2^{i-1}) = \alpha\sum_{i=0}^{k} 2^{k-i}2^{i\beta} + 6\gamma\sum_{i=0}^{k} 2^{k-i}2^{(i-1)\beta} =$$

$$(\alpha 2^k + 6\gamma 2^{k-\beta})\sum_{i=0}^{k} 2^{i(\beta-1)} = (\alpha + 6\gamma 2^{-\beta})\frac{2^\beta n^\beta - 2n}{2^\beta - 2} = \frac{\gamma 6(2^\beta - 3)(n^\beta - \frac{2n}{2^\beta})}{(2^\beta - 4)(2^\beta - 2)}$$

Conclusion

Thus, presented algorithm has the complexity of matrix multiplication. If we apply the standard matrix multiplication ($2n^3$ operations for the matrix $n \times n$), then we need only $\approx 2.5n^3$ operations.

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Control of matrix computations on distributed memory

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Abstract. Dedicated to research in the field of parallel computer algebra, in particular the parallelization of matrix recursive algorithms on a cluster with distributed memory. A new dynamic control scheme for matrix recursive algorithms is proposed. We considered in detail new software objects that ensure the effective operation of the dynamic control scheme.

Introduction

The first approach to creating parallel programs was a centralized dynamic LLP control scheme, in which one of the cluster nodes acted as the dispatcher for the entire computational process.

Next, the DDP scheme of the decentralized control was developed. In this scheme, each process node created its own dispatch process. However, in this scheme, there was no control over the depth of recruitment and the ability to switch to a new task until the current task was completed.

The new control scheme is called DAP-VAT-schemes. It differs in that it sequentially expands functions in depth, retaining all states at any nesting level until all calculations in the current computational subtree are completed. This allows any processor to freely switch from one subtask to another, without waiting for the completion of the current subtask account.

1. Recursive algorithm graph

1.1. Examples: matrix multiplication and triangular matrix inversion

For example, we present two simple block-recursive algorithms. Each of them contains a small number of types of recursive blocks.

The first algorithm is the calculation of the inverse matrix for a triangular non-degenerate matrix. After dividing the matrices into blocks, we get the equations

$$A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} x & 0 \\ z & k \end{pmatrix}$$

$$x = a^{-1}, \ k = d^{-1}, \ cx + dz = 0, \ z = -kcx.$$

The second algorithm is the algorithm of recursive block matrix multiplication $AB = C$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} l & m \\ n & p \end{pmatrix} = \begin{pmatrix} w1 & w2 \\ w3 & w4 \end{pmatrix}$$

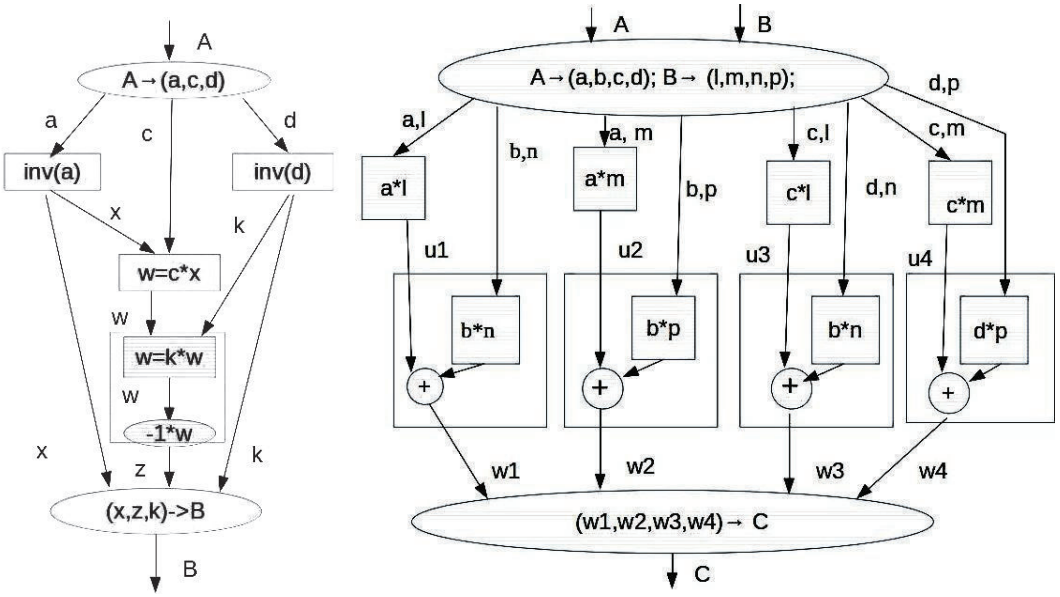

FIGURE 1. The graph of the recursive inversion algorithm of the triangular matrix (on the left) and the graph of the block-recursive matrix multiplication algorithm (on the right).

2. Computational control mechanism

Consider the components of the control mechanism of the computational process.

2.1. Drop

We divide the computational graph into separate compact subgraphs (drops). In Figure 1, the vertices, which are combined into one drop, are outlined in a square outline.

Thus, we define the drops as the smallest components of the computational graph that can be transferred to other processors.

2.2. Amine

Before the drop is calculated, we need to expand the corresponding subgraph. This subgraph is called amine. This amine also consists of drops.

For example, the amine $A \cdot B$ consists of 4 drops $A \cdot B$, 4 drops $A \cdot B + C$, one input and one output function.

2.3. Pine

All amines that are formed in one processor are stored in the general list, which is called Pine.

2.4. Vokzal

At the Vokzal are all the drop-tasks that are awaiting their direction to the calculations. These tasks are located at different levels. These levels correspond to the depth of recursion for drops.

2.5. Aerodrome

Each processor that sent a drop task is called a parent. The list of all parent processors is called an Aerodrome.

2.6. Terminal

The terminal is used to communicate with the child processors that were sent drop-tasks. All child processors are stored in the terminal.

3. Primary fields and functions

3.1. The main fields of the drop object

— PAD (np, na, nd)– address of this drop.
— Type — drop type (unique number in the list of all drop types).
— InData outData — these are vectors for input and output.
— Amine — the amine of this drop.
— RecNum — recursion number of the drop.
— Arcs — amine graph topology.

3.2. The main fields of the amine object

— PAD (np, na, nd) — address to return the result of the calculation of this drop.
— Type, inData, outData — the same as the drop.
— Drop — an array of all drops of a given amine.

4. Organization of computational threads

We use two threads: a computational thread and a dispatcher thread. These threads will run on each cluster processor.

4.1. CalcThread

The CalcThread waits for the arrival of the first drop task at the vokzal and starts the corresponding calculations.

4.1.1. CalcThread objects:

— Pine — list of amines on this processor.
— Vokzal — an array of lists of available drop tasks.
— Aerodrome — list of parent processors.
— Terminal — an array of child processor lists.
— CurrentDrop — current drop, which is calculated.

4.1.2. CalcThread functions:

— WriteResultsToAmin — the results of a drop calculation are written to its amine in the input data vectors of other drops.
— InputDataToAmin — create an amine from a drop, if a new task arrives, we make an input function.
— WriteResultsAfterInpFunc — write the result of the input function to all the amine drops.
— runCalcThread() — If a drop-result has come, we register it in another drop by topology (writeResultsToAmin). If additional components arrived, we make the input function and write to the amine its result. (inputDataToAmin & writeResultsAfterInpFunc) If a new drop arrives with a task, we look at the size of the input data. If the task is a leaf, it make a sequential calculation and it write the result to other drops (writeResultsToAmin). Otherwise, expand the amine (inputDataToAmin). If Vokzal is empty, then the isNotEmptyVokzal flag is set and the counting thread goes to wait another drop.

4.2. Dispatching Thread

The work of the dispatching thread can be divided into 10 processes:
— Waiting for completion signal.
— Reception task.
— Receive free processors.
— Receive and record the status of the child processor.
— Receive the result of the calculated drop and record these results in the corresponding amine.
— Receive non-main components and record it in the right place.
— Sending available tasks to free processors (if there are tasks and processors).
— Sending free processors to a child (if there are no drop tasks available, but there are free and child processors).
— Sending the entire list of free processors to the parent processor (if the Vokzal is empty and the Terminal does not contain child processors with positive levels).
— Sending drop results to parent processors.
— Sending additional components to child processors.

Conclusion

We gave a description of the universal dynamic paralleling scheme for recursive algorithms on the distributed memory cluster, described the main objects, their fields and functions, and also explained the operation of the two-thread system that runs on each cluster core. We have described six new objects that provide such a control mechanism and give the name of this scheme: drop, amine, pine, vokzal, aerodrome, terminal. This scheme can be applied to any matrix recursive algorithms, both with dense and sparse matrices. The scheme was implemented in the Java programming language using the OpenMPI and MathPartner [6] packages, and its work was tested on the above matrix multiplication and inversion algorithms. We plan to conduct a detailed experimental study of the effectiveness of this scheme on other recursive matrix algorithms.

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Weak Involutive bases over effective rings

Michela Ceria and Teo Mora

Abstract. We discuss the problems related with the extension of Janet theory to effectively given rings.

As remarked in 1992 by Schwartz [21], in 1920 after a cohoperation with Hilbert, Janet [11] introduced, under the name of complete/involutive bases both the notion of Gröbner bases and a computational algorithm which essentially anticipated Buchberger's [1, 2] Algorithm[1] (apparently in the strongest formulation given by Moller's Lifting Theorem [14]).
The recent extension of Buchberger Theory and Algorithm on each $\mathcal{R}$-module $\mathcal{A}$ [15, IV.50] [17, 5], where both $\mathcal{R}$ and $\mathcal{A}$ are assumed to be effectively given through their Zacharias representation [16] suggested us to investigate how far Janet's approach can be extended to more exotic settings. Clearly the combinatorial aspects of Janet completion necessarily require at least that, using the terminology of [15, IV.50], the associated graded ring $\mathcal{G}$ of $\mathcal{A}$ is an Ore-like extension [13, 6]; an interesting class of such rings, much wider than solvable polynomial rings [12] on which Seiler [22] applied Janet approach, has been recently proposed [18]: $\mathcal{A} = \mathcal{R}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle/\mathcal{I}, \mathcal{I} = \mathbb{I}(G)$ with

$$\begin{aligned} G &= \{X_jX_i - a_{ij}X_iX_j - d_{ij} : 1 \le i < j \le n\} \\ &\cup \{Y_lX_j - b_{jl}v_{jl}X_jY_l - e_{jl} : 1 \le j \le n, 1 \le l \le m\} \\ &\cup \{Y_kY_l - c_{lk}Y_lY_k - f_{lk} : 1 \le l < k \le m\} \end{aligned}$$

a Gröbner basis of $\mathcal{I}$ with respect to the lexicographical ordering $<$ on $\Gamma := \{X_1^{d_1}\cdots X_n^{d_n}Y_1^{e_1}\cdots Y_m^{e_m} | (d_1, \ldots, d_n, e_1, \ldots, e_m) \in \mathbb{N}^{n+m}\}$ induced by $X_1 < \ldots < X_n < Y_1 < \ldots < Y_m$ where

- a_{ij}, b_{jl}, c_{lk} are invertible elements in $\mathcal{R}$,
- $v_{jl} \in \{X_1^{d_1}\cdots X_j^{d_j} \mid (d_1, \ldots, d_j) \in \mathbb{N}^j\}$
- $d_{ij}, e_{jl}, f_{lk} \in \mathcal{A}$ with $\mathbf{T}(d_{ij}) < X_iX_j, \quad \mathbf{T}(e_{jl}) < X_jY_l, \quad \mathbf{T}(f_{lk}) < Y_kY_l.$

[1] Up to Second Buchberger Criterion [3] but probably including the other criteria proposed by Gebauer and Möller [8].

The associated graded ring $\mathcal{G}$ is obtained by setting $d_{ij} = e_{jl} = f_{lk} = 0$. We immediately remark that, unless we restrict to the case in which each $v_{jl} = \mathbf{1}_{\mathcal{A}}$, noetherianity is not sufficient to grant temination and finiteness.

Example 1. Simply consider Tamari's [23] ring $\mathbb{Q}\langle X, Y\rangle/\mathbb{I}(YX - X^2Y)$ where the principal ideal $\mathcal{I} = (X)$ has the infinite involutive basis $\{X^{2^i}Y^i, i \in \mathbb{N}\}$ each element having X as multiplicative variable.

Under this restriction, we obtain in any case a class of rings larger than solvable polynomial rings[2] even if $\mathcal{R}$ is assumed to be a field; there are in fact

- for each term $\tau \in \Gamma$ an automorphism $\alpha_\tau : \mathcal{R} \to \mathcal{R}$ and
- for each two terms $\tau_1, \tau_2 \in \Gamma$ an element $\varpi(\tau_2, \tau_1) \in \mathcal{R}$ so that the multiplicative $*$ arithmetic of $\mathcal{G}$ is defined by distributing the monomial product

$$a_1\tau_1 * a_2\tau_2 = a_i\alpha_{\tau_1}(a_2)\varpi(\tau_1, \tau_2)\tau_1 \circ \tau_2$$

where $\circ$ denote the classical multiplication in Γ.

Already under this restriction and even assuming $\mathcal{R}$ to be a field, the classical

Theorem 2. [9, Th.4.10] [10, Th.2.10] *If an involutive division is left(/right/restricted) continuous then left(/right/restricted) local involutivity of any set U implies its left(/right/restricted) involutivity.*

is not obvious [7]: it can be proved by means of Jacobi-like formulas which can be deduced on effective rings via associativity. The main problem arises when the coefficient ring $\mathcal{D}$, on which $\mathcal{R} = \mathcal{D}\langle\overline{\mathbf{v}}\rangle/I$ is a module, is not a field but just a PID[3]; as it was remarked by Seiler [22] one needs at least to follow the standard approach in Buchberger Theory and speak of *weak* and *strong* bases.

Example 3. [20] In the ideal $\mathcal{I} := \mathbb{I}(g_1, g_2) \subset \mathbb{Z}[X, Y], g_1 := 3X, g_2 := 2Y$, it holds $\mathcal{I} \ni g_3 := XY = g_1Y - g_2X$ while $3X \nmid XY$ and $2Y \nmid XY$.

As a consequence the characterization of a set U to be *involutive/complete with respect to an involutive division L* which in the field case [9, Def.4.1] [10, Def.2.4] simply requires that $\cup_{u\in U} u\,L(u, U) = \cup_{u\in U} u\,\Gamma \subset \Gamma$ must be reconsidered since we should require a formulation $\cup_{u\in U} u\,L(u, U) = \cup_{u\in U} u\,\mathsf{M}(\mathcal{A}) \subset \mathsf{M}(\mathcal{A}) := \{ct : t \in \Gamma, c \in \mathcal{R} \setminus \{0\}\}$ but, in general $\mathcal{N} := \cup_{u\in U} u\,\mathsf{M}(\mathcal{A}) \subsetneq \mathbb{I}(U) \cap \mathsf{M}(\mathcal{A}) = \mathrm{Span}_{\mathcal{R}}\{\mathcal{N}\} \cap \mathsf{M}(\mathcal{A})$.
For the moment we have postponed the investigation of the *strong* case and we [7] have adapted the terminology from the *terms* Γ with coefficients over a field to the *monomials* $\mathsf{M}(\mathcal{A})$, the coefficients being over an effectively given ring $\mathcal{R}$ and applied *Weispfenning multiplication* [24, 5] in order to deduce twosided (and subbilateral) bases from restricted ones, but mainly we have considered only the easiest *weak* case. In this setting, of course, we loose one stength of involutiveness, namely that any monomial $w \in \mathsf{M}(\mathcal{A})$ has at most one $L-$involutive divisor in U,

[2] where each α_τ is the identity and each $\varpi(\tau_2, \tau_1) = 1$ so that $a_1\tau_1 * a_2\tau_2 = a_1a_2\tau_1 \circ \tau_2$.
[3] the PIR case simply requires to deal with proper annihilators.

a property which can be granted, via *strong* bases, only when $\mathcal{R}$ itself is a PIR. Therefore reduction of a monomial $c\tau \in \mathsf{M}(\mathcal{A})$ must be performed considering all potential divisors $c_i\tau_i \in U$ such that $\tau_i \mid \tau$, $\tau = \upsilon_i \circ \tau_i$ and looking for relations $c = \sum_i a_i \alpha_{\upsilon_i} \varpi(\upsilon_i, \tau_i)$ and reduction be performed via classical Buchberger reduction.

In the *strong* cases, on the basis of [20, 14, 19], we guess that the test/completion for involutivity of a continuous involutive division, which in the field case (Theorem 2) is local involutivity, should be reformulated as

Claim 4. [10, Th.6.5] *Let L be a continuous involutive division. A polynomial set F is strong L-involutive if*

- *for each $f \in F$ and each non-multiplicative variable $x \in NM_L(lc(f), lc(F))$, the related J-prolongation $f \cdot x_i$,*
- *for each $f, g \in F$ the related P-prolongation $s\frac{lcm(\mathbf{T}(f),\mathbf{T}(g))}{\mathbf{T}(f)} f + t\frac{lcm(\mathbf{T}(g)g,\mathbf{T}(g))}{\mathbf{T}(f)}$, where c, s are the Bezout values such that $slc(f) + tlc(g) = \gcd(lc(f), lc(g))$,*
- *for each $f \in F$ the related A-prolongation af, a being the annihilator of $lc(f)$*

reduce all of them to zero modulo F.

There is still some research required in the strong case when $\mathcal{R}$ itself is PID; we need to investigate whether both the classical [9, 10] approach and the recent RID [4] suggestion are able to recover the division structure of polynomial domains.

Example 5. For the ideal $\mathcal{I} := \mathbb{I}\left(8X, 4X^3, 2X^6, 36Y^2, 6Y^3, Y^4\right) \subset \mathbb{Z}[X, Y]$ a (minimal) strong Gröbner basis is $\bar{U} := \{8X, 4X^3, 2X^6, 36Y^2, 4XY^2, 6Y^3, 2XY^3, Y^4\}$; with respect the Janet/Pommaret division a strong minimal involutive basis is

$$\begin{aligned}\tilde{U} &:= \left\{8X^{1+i}Y^j, 0 \le i \le 1, 0 \le j \le 1\right\} \cup \left\{4X^{3+i}Y^j, 0 \le i \le 2, 0 \le j \le 1\right\}\\ \cup &\quad \left\{2X^6Y^j, 0 \le j \le 3\right\} \cup \left\{36Y^2, 6Y^3, Y^4\right\} \cup \left\{4X^{1+i}Y^2, 0 \le i \le 4\right\} \cup \left\{2X^{1+i}Y^3, 0 \le i \le 4\right\}\end{aligned}$$

with

τ	$M(\tau)$	$NM(\tau)$
Y^4	$\{X, Y\}$	$\emptyset$
$\left\{2X^6Y^j, 0 \le j \le 3\right\}$	$\{X\}$	$\{Y\}$
$\emptyset$	$\{Y\}$	$\{X\}$
$\tilde{U} \setminus \{2X^6, 2X^6Y, 2X^6Y^2, 2X^6Y^3, Y^4\}$	$\emptyset$	$\{X, Y\}$

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Numerical Symbolic Dynamics: Complexity of Finite Sequences

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Abstract. We study the complexity of the finite sequences that were constructed numerically by integrating equations of motion of the equal mass free-fall three-body problem. We construct symbolic sequences using close binary approaches, in which the corresponding symbol in the sequence is the number of the distant body. Different approaches to estimate complexity are considered: Shannon entropy, Kolmogorov complexity and Arnold complexity.

Introduction

Description of the problem and some history can be found in [6] and [7]. See also the paper by Chase et al. [3] in this issue. Here, we concentrate on the analysis of complexity of finite sequences and compare different methods to analyze complexity. Shannon entropy is one standard method to estimate "randomness" of the sequence. We also use other approaches; for instance, we use the length of the archive as an estimate of the Kolmogorov complexity. Another approach to estimate complexity of the finite (binary) sequences was suggested by Arnold [2]. This method is based on the first differences of the sequences. We compare the results obtained via the different methods.

The equal mass free-fall three-body problem is convenient for study since it allows easy visualization of initial configuration: if we place two bodies in the points $(-0.5; 0)$ and $(0.5; 0)$, then all possible configurations will be covered if we place the third body inside region D bounded by two straight line segments and with the arc of the unit circle centered at $(-0.5, 0)$ (Fig. 1) [1]. This region is used in the following visualizations.

We used symplectic code by Seppo Mikkola (Tuorla Observatory, University of Turku) [5] for numerical simulations.

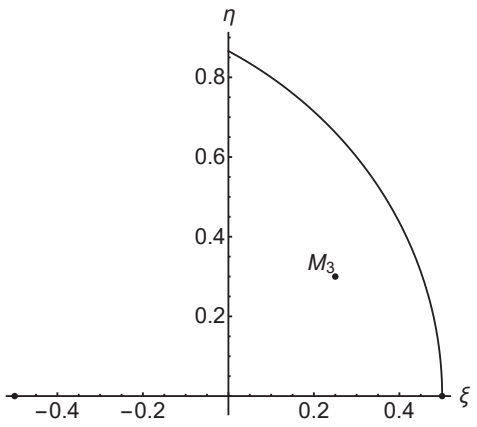

FIGURE 1. Region D of initial conditions.

1. Shannon entropy and Kolmogorov complexity

We construct symbolic sequences using binary encounters. We detect the minimum distance between two bodies, and the corresponding symbol is the number of the distant body. Some systems disrupt fast, so some sequences are short. In this case end of the sequence consists of 0's. Thus, our symbols are from the alphabet {0, 1, 2, 3}. Some systems have a long life (e.g. metastable systems [4]), so their corresponding sequences are long. To have a reasonable computing time, we constructed symbolic sequences of length 100. Since we are interested in the analysis of active three-body interactions, as one approach we consider sub-sequences of each of these sequences, increasing the length step-by-step, calculating Shannon entropy for each of these sub-sequences, and finding maximum value of these entropies. Maximum values (and moment of time/length of the sub-sequence) correspond to the stage of active interaction between bodies. Figure 2 shows maximum values of Shannon entropy.

Some features revealed on the histogram of maximum values of the entropy and seen on the scatterplot of maximum values of the entropy - corresponding length of symbolic sequence are studied by Chase et al. [3].

As an estimation of the Kolmogorov complexity we use the length of the archive of the sequence. Results are shown on Fig. 3.

2. Complexity of finite sequences

Arnold [2] suggested the following approach to estimate the complexity of finite binary sequences (or complexity of a function x, regarding j's element of the sequence x_j as a function of the argument j). Consider a set M of all possible sequences of length n. Let us define (following Newton's idea) the increment sequence: we thus consider the linear operator A: $M \to M$, $y = Ax$ defined by the formula $y_j = x_{j+1} - x_j$. To have n increments, we define $x_{n+1} = x_1$, making our sequence x cyclic (the function x, whose value at j is x_j, is then n-periodic).

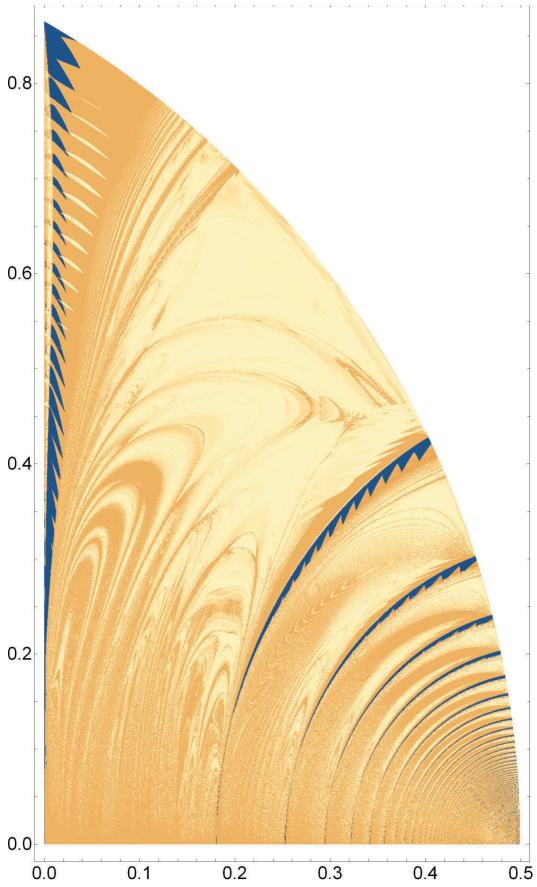

FIGURE 2. Maximum values of the entropy.

The map A of the finite set M into itself is described by a directed graph with 2^n vertices $x \in M$. In this graph, exactly one edge starts from each vertex x (and leads to Ax). It is convenient to denote binary sequences by corresponding binary numbers: sequence $x = (x_1, \ldots, x_n)$ corresponds to $X = x_1 \cdot 2^{n-1} + x_2 \cdot 2^{n-2} + \cdots + x_n \cdot 1$. Examples of these graphs are given in Figures 4 - 7. Each connected component of the graph contains a cycle, and it contains only one cycle [2]. Usually, there are several connected components of the type $(O_m * T_{2^k})$, where O_m is an m-vertex cycle, framed by a forest of m rooted (binary) trees T with 2^k vertices directed to the roots belonging to the cycle O_m, see Figures 4 - 7. An exception is the cases when n is a power of 2, in these cases there is only one connected component, and all sequences converge to 0 (see Fig. 5 and Fig. 7 left).

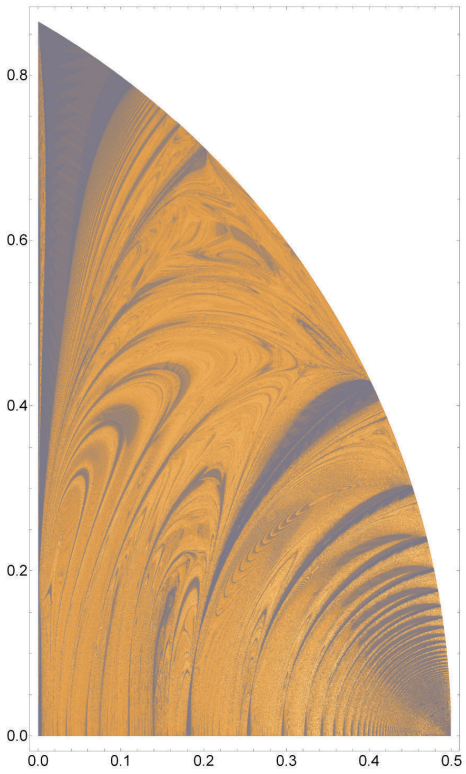

FIGURE 3. Kolmogorov complexity estimated as a length of the archive of the symbolic sequence.

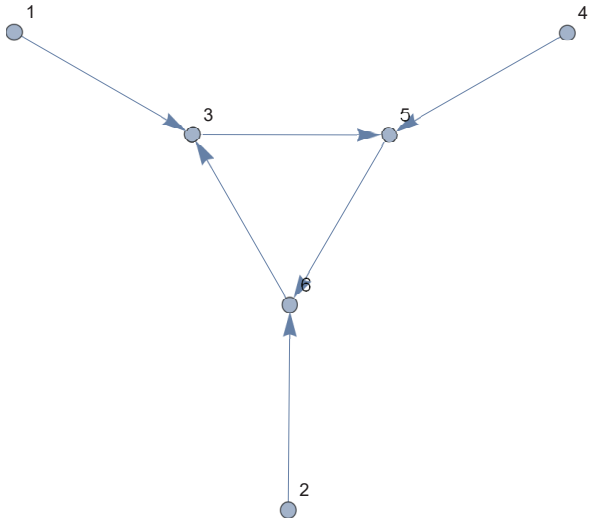

FIGURE 4. Graph of the map A: $M \to M$ for $n = 3$.

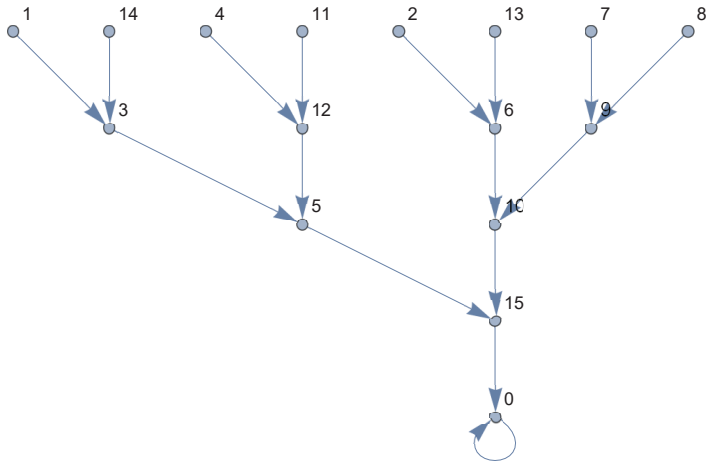

FIGURE 5. Graph of the map A: $M \to M$ for $n = 4$.

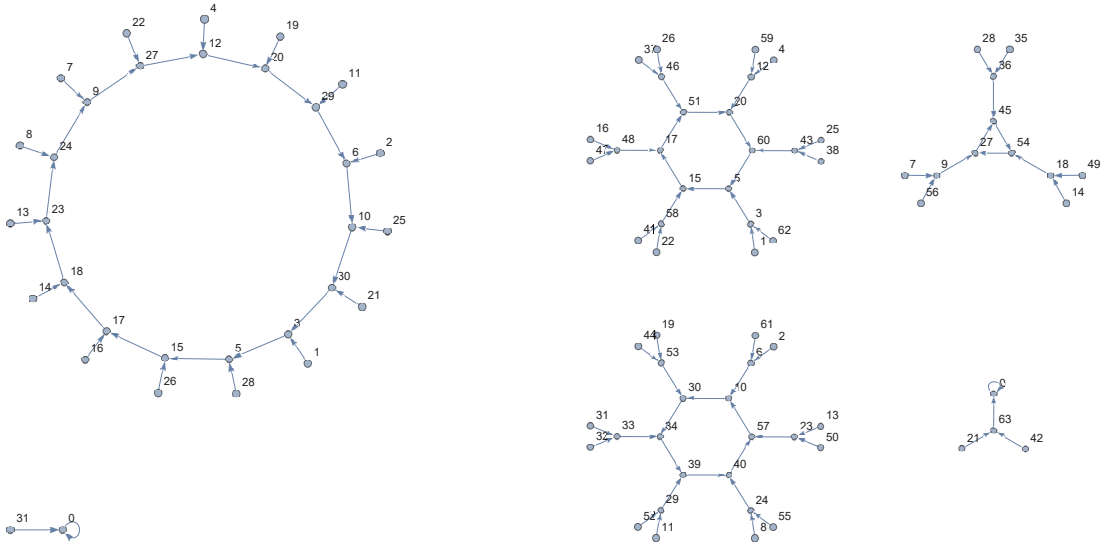

FIGURE 6. Graph of the map A: $M \to M$ for $n = 5$ (left) and $n = 6$ (right).

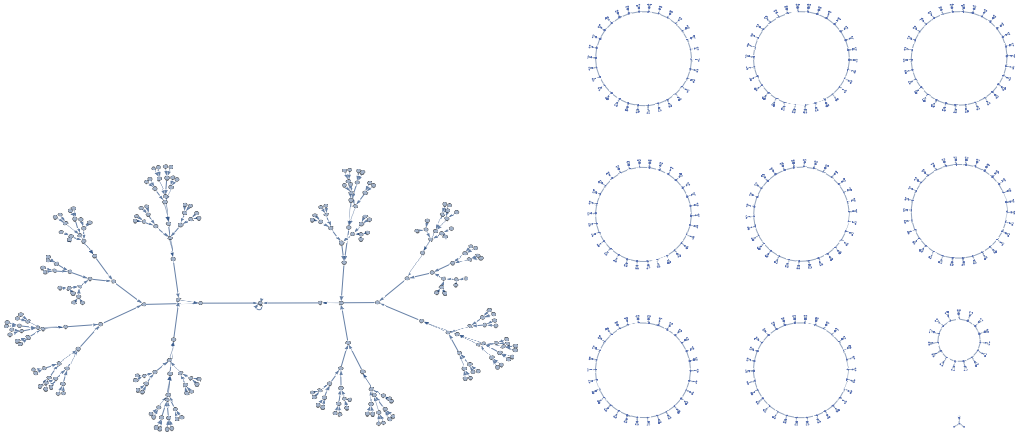

FIGURE 7. Graph of the map A: $M \to M$ for $n = 8$ (left) $n = 10$ (right).

The definition of the complexity is as follows: we say that an object x is more complicated if the length of the cycle of the component of the graph containing the point x is larger. Inside the components whose cycles have equal lengths, a vertex is said to be more complicated if its distance from the cycle is larger [2].

In our case, we deal with sequences from the alphabet {0, 1, 2, 3}, so our sequences are quaternary. Some examples of analogs of Figures 4 - 7 for quaternary sequences are shown in Figures 8 - 9.

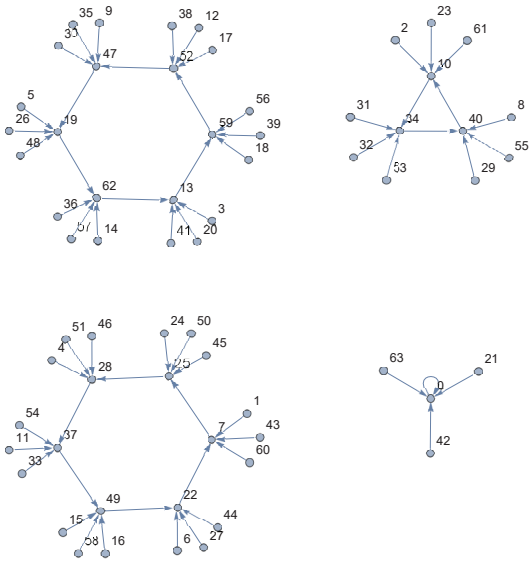

FIGURE 8. Graph of the map A: $M \to M$ for quaternary sequences, $n = 3$.

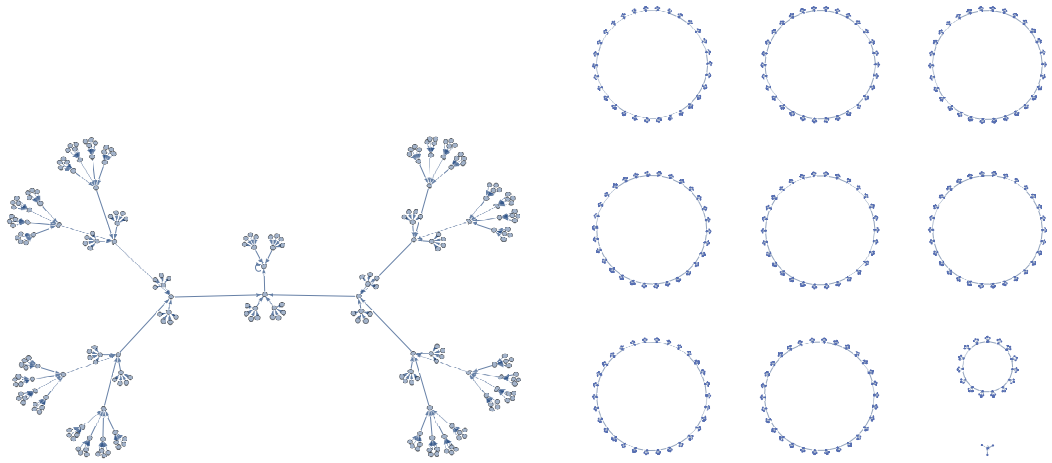

FIGURE 9. Graph of the map A: $M \to M$ for quaternary sequences, $n = 4$ (left) and $n = 5$ (right).

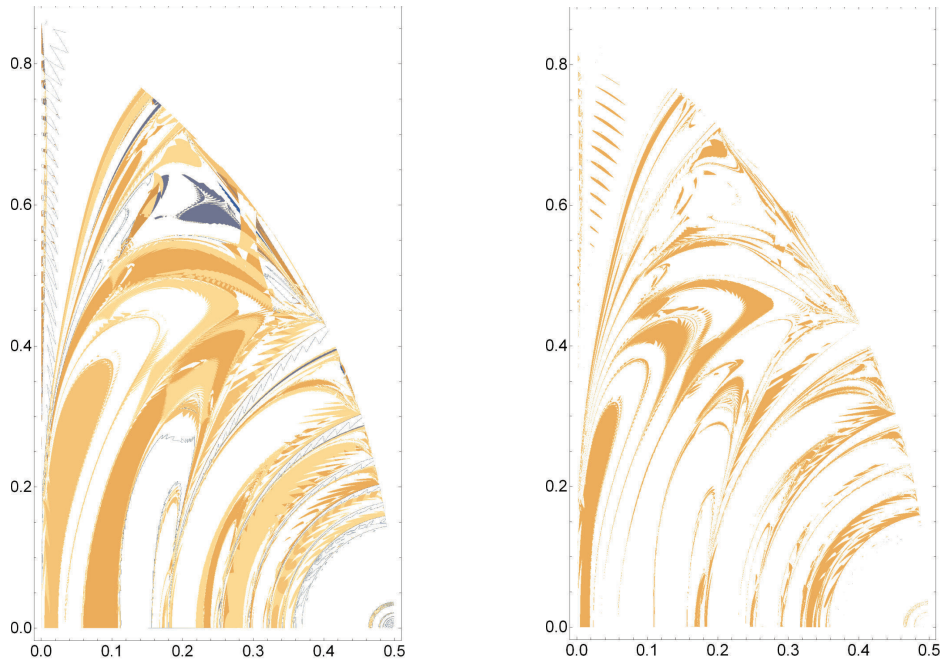

FIGURE 10. Arnold complexity of the symbolic sequences for $n = 8$ (left) and $n = 9$ (right).

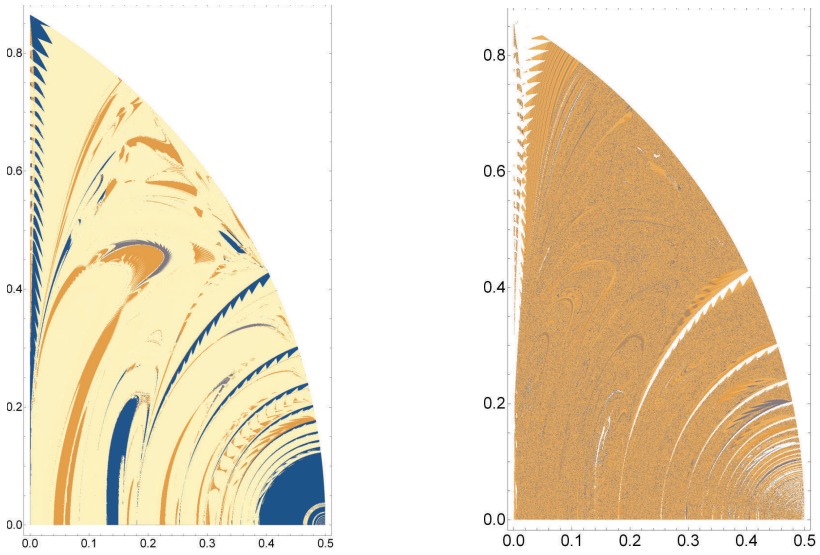

FIGURE 11. Arnold complexity of the symbolic sequences for $n = 10$ (left) and $n = 64$ (right).

Unfortunately, studying long sequences puts a heavy demand on computer hardware: the size of the arrays is 4^n, so we considered only cases $n = 8 \ldots 13, 16, 32$ and 64. Some examples are given in figures 10 - 12. Using $n = 2^k$ is possible for larger values of n since in this case the corresponding graph is connected (tree) and all iterations converge to 0, see, e.g. Fig. 5.

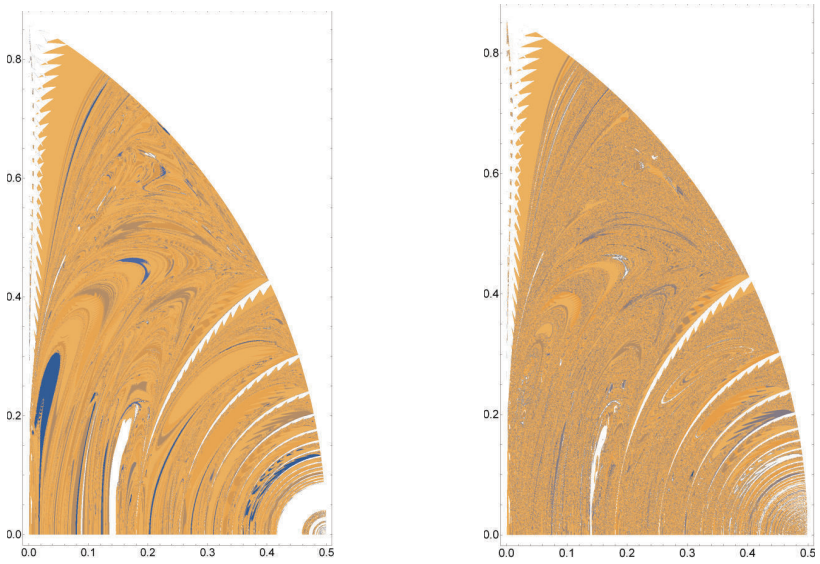

FIGURE 12. Arnold complexity of the symbolic sequences for $n = 16$ (left) and $n = 32$ (right).

3. Discussion

We have used different methods to estimate the complexity of finite symbolic sequences that were obtained by numerical integration of the equations of free-fall equal-mass three-body problem. These methods stress different features of the sequences studied, but different methods of analysis of the complexity, and even different ways of constructing symbolic sequences resulted in similar partitioning of the phase space.

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Litvinov-Maslov Dequantization of Matrix Algebras: New Insights and Techniques

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Abstract. Tropical mathematics studies semifields with idempotent addition obtained via extreme logarithmic deformation of the real semifield known as Litvinov-Maslov dequantization. We investigate unobvious aspects of its generalization to the matrix case giving raise to a new class of tropical algebras that we refer to as uncanonical. We classify tropical matrix algebras obtained by dequatization of the real one and provide computational examples.

Introduction

Tropical mathematics studies semirings and semifields with idempotent addition. For example, the semifield $\mathbb{R}_{\max,+}$ is defined as the set $\mathbb{R} \cup \{-\infty\}$ equipped with addition $a \oplus b = \max(a, b)$ and multiplication $a \otimes b = a + b$, zero $\mathbb{0} = -\infty$ and unit $\mathbb{1} = 0$. The relationship between usual and tropical algebras was described by G.L. Litvinov and V.P. Maslov in terms of logarithmic deformation of the real semifield (quantization) parametrized by some paramter $h \in \mathbb{R}$ and the subsequent taking the limit when $h \to 0$ (dequantization). Other tropical algebras like $\mathbb{R}_{\max,\times}$, $\mathbb{R}_{\min,+}$ and $\mathbb{R}_{\min,\times}$ could be obtained via the same procedure depending on how exactly the original algebra is being deformed and how the limit is being taken [1].

A semimodule over a semifield is a generalization of the classical notion of a linear space over a field, wherein the corresponding scalars are the elements of a given semiring and a multiplication is defined of the ring and elements of the module [2, 3]. The matrix semimodule could be turned into matrix algebra considering only square matrices semimodule with respect to matrix multiplication. The tropical matrix algebra has two important but not equivalent constructions through Litvinov-Maslov dequationtization of real matrix. The most remarkable fact is that one of them admits one additional way of dequantization giving raise to a new classes of tropical matrix algebras that we refer to as uncanonical. We classify tropical matrix algebras obtained by dequatization of the real one and provide computational examples.

1. Dequantization of Scalar Algebras

A semifield $(\mathbb{S}, \oplus, \odot, \mathbb{0}, \mathbb{1})$ is a set $\mathbb{S}$ equipped with addition $\oplus$ and multiplication $\otimes$ operations, zero $\mathbb{0}$ and unit $\mathbb{1}$ elements such that $(\mathbb{S}, \oplus, \mathbb{0})$ is a commutative monoid, $(\mathbb{S}/\{\mathbb{0}\}, \otimes, \mathbb{1})$ is a comutative group, $\otimes$ is distributive over $\oplus$, $\mathbb{0}$ is an absorbing element $a \otimes \mathbb{0} = \mathbb{0}$. Tropical mathematics studies semifields with idempotent addition $a \oplus a = a$. Taking into account the multiplication group properties of semifilds division $a \oslash b = ab^{-1}$ is also available.

Algebra	$\mathbb{S}$	$a \oplus b$	$a \otimes b$
$\mathbb{R}_{+_h,\times}$	$\mathbb{R}_+ \cup \{h\ln(0)\}$	$h\ln(e^{\frac{a}{h}} + e^{\frac{b}{h}})$	$a \times b$
$\mathbb{R}_{+_h,\times_h}$	$\mathbb{R} \cup \{h\ln(0)\}$	$h\ln(e^{\frac{a}{h}} + e^{\frac{b}{h}})$	$a + b$

TABLE 1. Quantized scalar semirings

For example, $\mathbb{R}_{\max,+}$ is the set $\mathbb{R} \cup \{-\infty\}$ equipped with addition $a \oplus b = \max(a, b)$ and multiplication $a \otimes b = a + b$, zero $\mathbb{0} = -\infty$ and unit $\mathbb{1} = 0$. According to Litvinov-Maslov approach an explicit construction of tropical algebras listed in Table 2 is obtained via the composite map $\ell_h \circ \mu_h : \mathbb{R}_+ \to \mathbb{S}_h \to \mathbb{S}$ that sequentially transforms the real semifield $\mathbb{R}_+$ into a quantized semiring $\mathbb{S}_h$ by $\mu_h : \mathbb{R}_+ \to \mathbb{S}_h$ and then $\mathbb{S}_h$ into tropical semifields $\mathbb{S}$ by raking the corresponding limit $\ell_h : \mathbb{S}_h \to \mathbb{S}$.

Algebra	$\mathbb{S}$	$a \oplus b$	$a \otimes b$
$\mathbb{R}_{\min,\times}$	$\mathbb{R}_+ \cup \{+\infty\}$	$\min(a, b)$	$a \times b$
$\mathbb{R}_{\max,\times}$	$\mathbb{R}_+ \cup \{-\infty\}$	$\max(a, b)$	$a \times b$
$\mathbb{R}_{\min,+}$	$\mathbb{R} \cup \{+\infty\}$	$\min(a, b)$	$a + b$
$\mathbb{R}_{\max,+}$	$\mathbb{R} \cup \{-\infty\}$	$\max(a, b)$	$a + b$

TABLE 2. Tropical semirings

The role of $\mathbb{S}_h$ could be played by partially quantized $\mathbb{R}_{+_h,\times}$ or fully quantized $\mathbb{R}_{+_h,\times_h}$ semifield, where $+_h$ and $\times_h$ denote new operations induced by the variable change $x \to h\ln x$ defined as $a +_h b = \ln(\exp(a/h) + \exp(b/h))$ and $a \times_h b = \ln(\exp(a/h) \times \exp(b/h)) = a + b$ and parametrized by $h \in \mathbb{R}$ playing the role of Plank's constant. This parameter $h \in \mathbb{R}$ generates an ordered sequence of semirings $\mathbb{S}_h$ that has a limit depending on its sign defined in the following way

$$\mathbb{R}_{+_h,\times} = \begin{cases} \mathbb{R}_{\min,\times}, & h \to -0; \\ \mathbb{R}_{\max,\times}, & h \to +0; \end{cases} \qquad \mathbb{R}_{+_h,\times_h} = \begin{cases} \mathbb{R}_{\min,+}, & h \to -0; \\ \mathbb{R}_{\max,+}, & h \to +0. \end{cases}$$

2. Dequantization of Matrix Algebras

A semimodule over a semifield is a generalization of the notion of vector space over a field wherein the corresponding scalars are the elements of an arbitrary given

semiring and a multiplication is defined between elements of the semiring and elements of the semimodule. Matrix semimodule could be turned into an algebra by considering square matrices with respect to their multiplication. The tropical matrix algebra has two important but not equivalent constructions through Litvinov-Maslov dequationtization of real matrix. The most remarkable fact is that one of them admits one additional way of dequantization giving raise to a new classes of tropical matrix algebras that we refer to as uncanonical.

Algebra	$\mathbb{S}_h$	$a \oplus b$	$a \otimes b$
$\mathbb{R}_h^{\langle n\times n\rangle}$	$\big(\mathbb{R} \cup \{h\ln(0)\}\big)^{n\times n}$	$A_{ij} +_h B_{ij}$	$\sum_{k=1}^{n} A_{ik} \times B_{kj}$
$\mathbb{R}_h^{[n\times n]}$	$\big(\mathbb{R}_+ \cup \{h\ln(0)\}\big)^{n\times n}$	$A_{ij} +_h B_{ij}$	$\sum_{k=1}^{n} A_{ik} \times_h B_{kj}$
$\mathbb{R}_h^{(n\times n)}$	$\big(\mathbb{R} \cup \{h\ln(0)\}\big)^{n\times n}$	$A_{ij} +_h B_{ij}$	$\sum_{hk=1}^{n} A_{ik} \times_h B_{kj}$

TABLE 3. Quantized matrix semirings

Square matrix algebra over $\mathbb{S}$ is the set $\mathbb{S}^{n\times n}$ with respect to addition $\boxplus$ and multiplication $\boxtimes$, zero O and unit I defined for $A, B \in \mathbb{S}^{n\times n}$ by the formulas

$$\{A \boxplus B\}_{ij} = A_{ij} \otimes B_{ij}, \quad \{A \boxtimes B\}_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}.$$

In our notation matrix operations are denoted by box signs and scalar operations – by circle signs. So, the tropical matrix algebra $\mathbb{S}$ has two important but non-equivalent constructions: (1) quantization-dequantization of real scalars, and the subsequent modularization of tropical scalars, (2) modularization of real scalars,and the subsequent quantization-dequantization of real matrices. As in previous case the limits coulb be taken.

Algebra	$\mathbb{S}$	$\{A \oplus B\}_{ij}$	$\{A \otimes B\}_{ij}$
$\mathbb{R}_{\min}^{\langle n\times n\rangle}$	$\big(\mathbb{R} \cup \{+\infty\}\big)^{n\times n}$	$\min(A_{ij}, B_{ij})$	$\sum_{k=1}^{n} A_{ik} \times B_{kj}$
$\mathbb{R}_{\max}^{\langle n\times n\rangle}$	$\big(\mathbb{R} \cup \{-\infty\}\big)^{n\times n}$	$\max(A_{ij}, B_{ij})$	$\sum_{k=1}^{n} A_{ik} \times B_{kj}$
$\mathbb{R}_{\min}^{[n\times n]}$	$\big(\mathbb{R} \cup \{+\infty\}\big)^{n\times n}$	$\min(A_{ij}, B_{ij})$	$\min_{k=1}^{n} A_{ik} \times B_{kj}$
$\mathbb{R}_{\max}^{[n\times n]}$	$\big(\mathbb{R} \cup \{-\infty\}\big)^{n\times n}$	$\max(A_{ij}, B_{ij})$	$\max_{k=1}^{n} A_{ik} \times B_{kj}$
$\mathbb{R}_{\min}^{(n\times n)}$	$\big(\mathbb{R} \cup \{+\infty\}\big)^{n\times n}$	$\min(A_{ij}, B_{ij})$	$\min_{k=1}^{n} A_{ik} + B_{kj}$
$\mathbb{R}_{\max}^{(n\times n)}$	$\big(\mathbb{R} \cup \{-\infty\}\big)^{n\times n}$	$\max(A_{ij}, B_{ij})$	$\max_{k=1}^{n} A_{ik} + B_{kj}$

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Foliation of the special linear group $SL(n, \mathbb{R})$ into conjugacy classes from polynomial algebra point of view

Yuri Palii

Введение

The foliation of a Lie group G under the conjugacy action as well as the coadjoint orbit foliation of dual space $\mathfrak{g}^*$ of a Lie algebra are interesting objects both for mathematicians and physicists. The natural choice of matrix elements as parameters for matrix groups allows to apply algebraic methods for local trivialization of these foliations. To find foliated coordinates (more precisely, parameters on the conjugacy classes or on the coadjoint orbits), one should solve a system of partial differential equations which expresses their invariance under the transformations related to the transversal vector fields. The components of vector fields are minors of the group element (algebra element). Keeping this in mind we look for the foliated coordinates as rational functions of the matrix elements. In this paper we show how to reduce this problem to solution of a linear system of equations.

Key words: Lie groups, Lie algebras, conjugacy classes, coadjoint orbits, foliation, Lie derivative, invariants

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A universality theorem for stressable graphs in the plane

Gaiane Panina

Universality theorems (in the sense of N. Mnëv) claim that the realization space of a combinatorial object (a point configuration, a hyperplane arrangement, a convex polytope, etc.) can be arbitrarily complicated. In the paper, we prove a universality theorem for a graph in the plane with a collection of signs of its possible equilibrium stresses ("oriented matroid of stresses").

This research is motivated by the Grassmanian stratification (Gelfand, Goresky, MacPherson, Serganova) and a recent series of papers on stratifications of configuration spaces of tensegrities (Doray, Karpenkov, Schepers, Servatius).

Here are details: let $\Gamma = (V, E)$ be a graph without loops and multiple edges, where $V = \{v_1, ..., v_m\}$ is the set of vertices, and E is the set of edges. A *realization* of Γ is a map $p : V \to \mathbb{R}^2$ such that $(ij) \in E$ implies $p(v_i) \neq p(v_j)$. We abbreviate $p(v_i)$ as p_i.

A *stress* $\mathfrak{s}$ on a realization (Γ, p) is an assignment of real scalars $\mathfrak{s}(i, j)$ to the edges. One imagines that each edge is turned to a (either compressed or extended) spring. A stress $\mathfrak{s}$ is called a *self-stress*, or an *equilibrium stress*, if at every vertex p_i, the sum of the forces produced by the springs vanishes:

$$\sum_{(ij)\in E} \mathfrak{s}(i, j)\mathbf{u}_{ij} = 0.$$

Here $\mathbf{u}_{ij} = \frac{p_i - p_j}{|p_i - p_j|}$ is the unit vector pointing from p_j to p_i.

Given realization (Γ, p), the set of all self-stresses $\mathfrak{S}(\Gamma, p)$ is a linear space which naturally embeds in $\mathbb{R}^e$, where $e = |E|$. Set $\mathcal{M}(\Gamma, p) := SIGN(\mathfrak{S}(\Gamma, p))$. In simple words, we do the following: enumerate somehow the edges of the graph, and for each non-trivial stress, list the signs of its values on all the edges. We obtain a collection of strings (elements of $(+, -, 0)^e$).

Given a graph Γ and an oriented matroid $\mathcal{M}$, define the *realization space* of the pair $(\Gamma, \mathcal{M})$ as the space of all realizations of Γ that yield the oriented matroid $\mathcal{M}$. We factorize the space by the action of the general linear group:

$$\mathcal{R}(\Gamma, \mathcal{M}) = \{p : \mathcal{M}(\Gamma, p) = \mathcal{M}\}/GL(2).$$

In general, semialgebraic sets are subsets of some Euclidean space $\mathbb{R}^N$ defined by polynomial equations and inequalities. A semialgebraic set is called a *fat basic primary semialgebraic set (FBP semialgebraic set)* if there are no defining equations, all the defining inequalities are strict, and the coefficients of all the defining polynomials are rational.

Our **main result** is: For each FBP semialgebraic set A, there exists a graph Γ and an oriented matroid $\mathcal{M}$ such that the realization space $\mathcal{R}(\Gamma, \mathcal{M})$ is stably equivalent to A.

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Counting perfect matching with a selected edge on $C_m \times C_n$ tori

S.N. Perepechko

Introduction

In this report, we discuss the dimer problem, which is one of the classical lattice models of statistical mechanics. The combinatorial interpretation of this model is reduced to the enumeration of close-packed dimer configurations. For many kinds of graphs popular in physical applications, such configurations are perfect matchings.

For qualitative estimation of the model parameters, regular graphs of degree q are usually used, assuming that the probability of belonging of each edge of the lattice to the matching is the same and is equal to $1/q$. In the thermodynamic limit, this approximation comes to a satisfactory description of the properties of the model, but the conditions for its use are not entirely clear.

In our study, the tori $C_m \times C_n$ of even order are used as an example of regular graphs. The set of edges of these graphs can be divided into subsets of E_m and E_n. The edge $e_m \in E_m$ if it belongs to one of the simple cycles C_m. Accordingly, the edge $e_n \in E_n$ if it belongs to one of the simple cycles C_n.

During the study, one of the edges $e_m \in E_m$ was selected and the number of perfect matchings in the graph containing this edge was found. Denote the obtained value by $K^{(m)}_{m,n}$. The tori $C_m \times C_n$ are convenient because $K^{(m)}_{m,n}$ does not depend on the choice of the edge e_m.

Having performed similar calculations for one of the edges e_n, one can find the value of $K^{(n)}_{m,n}$ and estimate the occupation probabilities of the e_m and e_n edges

$$R^{(m)}_{m,n} = \frac{K^{(m)}_{m,n}}{K_{m,n}}, \quad R^{(n)}_{m,n} = \frac{K^{(n)}_{m,n}}{K_{m,n}},$$

where $K_{m,n}$ is the total number of perfect matchings on the torus. In the present work, for small values of $m < 11$, closed form expressions for $R^{(m)}_{m,n}$ and $R^{(n)}_{m,n}$ are found and their asymptotic behavior is investigated. In particular, explicit values

for $R_m^{(m)} = \lim_{n\to\infty} R_{m,n}^{(m)}$ and $R_m^{(n)} = \lim_{n\to\infty} R_{m,n}^{(n)}$ are obtained. For a few values of $m > 10$, extensive sets of numeric data were computed. The use of convergence acceleration methods allowed to estimate $R_m^{(m)}$ and $R_m^{(n)}$ with sufficient accuracy for these values of m.

1. The technique of calculation

At present, simple methods for calculating $K_{m,n}^{(m)}$ and $K_{m,n}^{(n)}$ with arbitrary values of torus parameters are unknown. To derive closed form expressions, it was necessary to fix one of the torus parameters and solve a set of enumeration problems for each individual value $m > 2$. However, the approach used in this work has certain advantages. The fact is that a wide range of recurrence relations were obtained earlier [1], which are satisfied by the sequences $\{K_{m,n}\}$ at fixed values of $m < 21$. If the sequences $\{K_{m,n}^{(m)}\}$ and $\{K_{m,n}^{(n)}\}$ satisfy the same or similar recurrence relations, then the values of $K_{m,n}^{(m)}$ and $K_{m,n}^{(n)}$ can be expressed as linear combinations of $K_{m,n}$.

To enumerate perfect matchings, the method proposed half a century ago by Wilf was used [2]. Modern computer algebra systems allow its effective implementation not only for small graphs, but also for graphs of moderate order containing several thousand vertices [3].

In our work, all calculations were performed in the Maple system. The advantages of this system include the effective implementation of operations on integers of very large bit width, support for multithreading, as well as the ability to derive recurrence relations directly from the initial segments of numerical sequences. Solutions of recurrence relations were found using the `rsolve` procedure, and then simplified by the built-in tools of the system. Various manipulations of the recurrence relations are greatly facilitated when using commands from the `genfunc` package.

Since the orders of recurrence relations increase exponentially [1], their derivation turns out to be extremely resource-intensive. For this reason, for $m > 10$, only small initial segments of the $\{K_{m,n}^{(m)}\}$ and $\{K_{m,n}^{(n)}\}$ sequences were calculated. The length of these initial segments was chosen so as to ensure the necessary accuracy of finding the limit values $R_m^{(m)}$ and $R_m^{(n)}$.

The length of the initial segment can be significantly reduced if we use the methods of convergence acceleration. When choosing the acceleration method, we were guided by the fact that all the roots of the denominators of generating functions associated with the $\{K_{m,n}\}$ sequences are simple and real. In this case, it makes sense to use the well-known ε-algorithm [4]. The implementation of this algorithm in the computer algebra system allows all intermediate calculations to be performed in rational numbers, which eliminates rounding errors and instabilities inherent in all extrapolation methods.

Evaluation of the coefficients of the asymptotic expansions of $R_m^{(m)}$ and $R_m^{(n)}$ was carried out by numerical data fitting. The least squares method was used, implemented in the `LSSolve` procedure from the `Optimization` package.

Since the necessary condition for the existence perfect matching in the graph is the parity of its order, then for odd m, the parameter n must be even. In this case, it is advisable to somewhat change the notation made earlier. We assume that for odd m, the value of $K_{m,n}^{(m)}$ denotes the number of perfect matchings with the selected edge e_m in the graph $C_m \times C_{2n}$. The value of $K_{m,n}^{(n)}$ should be subjected to the same adjustments. As a result of the changes, the sequences $\{K_{m,n}^{(m)}\}$ and $\{K_{m,n}^{(n)}\}$ will not contain zero elements and the order of recurrence relations will be halved. Previously, similar notation was used in [1] when calculating $K_{m,n}$.

2. Some results

The results of the calculations showed that for all the studied values of m, the sequences $\{K_{m,n}^{(n)}\}$ and $\{K_{m,n}\}$ satisfied the same recurrence relations. The order of the recurrence relation for the sequence $\{K_{m,n}^{(m)}\}$ has always been one less for odd m and 2 less for even m. For example, for $m = 3$ we get

$$K_{3,n}^{(n)} = 6K_{3,n-1}^{(n)} - 6K_{3,n-2}^{(n)} + K_{3,n-3}^{(n)}, \quad K_{3,n}^{(m)} = 5K_{3,n-1}^{(m)} - K_{3,n-2}^{(m)}.$$

Taking into account the initial data $K_{3,2}^{(n)} = 15, K_{3,3}^{(n)} = 64, K_{3,4}^{(n)} = 299$ and $K_{3,2}^{(m)} = 10, K_{3,3}^{(m)} = 48$, the values of $K_{3,n}^{(m)}$ and $K_{3,n}^{(n)}$ can be expressed in terms of the total number of perfect matchings on the torus

$$K_{3,n}^{(m)} = \frac{1}{21}\left(K_{3,n+1} - K_{3,n-1}\right), \quad K_{3,n}^{(n)} = \frac{1}{2}K_{3,n} - K_{3,n}^{(m)}.$$

Based on the well-known formula for $K_{3,n}$ and the expressions obtained above, it is easy to verify the monotonicity of the sequences $\{R_{3,n}^{(n)}\}$ and $\{R_{3,n}^{(m)}\}$, with the sequence $\{R_{3,n}^{(n)}\}$ decreasing and $\{R_{3,n}^{(m)}\}$ increasing. The same condition was fulfilled for all other odd m. The limiting values of the edges e_m and e_n occupation probabilities have a simple form: $R_3^{(m)} = 1/\sqrt{21}$, $R_3^{(n)} = 1/2 - 1/\sqrt{21}$.

For even m, the relationship between the values of $R_{m,n}^{(m)}$ and $R_{m,n}^{(n)}$ is more complex. When $n < m$, the result of comparing these values depends on the parity of torus parameters. Obviously, for $n = m$, the edges e_m and e_n are equivalent, therefore, $R_{m,n}^{(m)} = R_{m,n}^{(n)}$. However, for $n > m$, perfect matchings contain more edges belonging to the cycles C_m, so $R_{m,n}^{(m)} > R_{m,n}^{(n)}$.

Due to the rapid growth of the orders of recurrence relations, the explicit formulas for $K_{m,n}^{(m)}$ and $K_{m,n}^{(n)}$ turn out to be cumbersome even for small m. For example, when $m = 4$, we obtain a recurrent relation of the sixth order, which is satisfied by the members of the sequence $\{K_{4,n}^{(m)}\}$

$$K_{4,n}^{(m)} = 4K_{4,n-1}^{(m)} + 5K_{4,n-2}^{(m)} - 24K_{4,n-3}^{(m)} + 5K_{4,n-4}^{(m)} + 4K_{4,n-5}^{(m)} - K_{4,n-6}^{(m)}.$$

To calculate $R_{4,n}^{(m)}$, it is convenient to express the values of $K_{4,n}^{(m)}$ in terms of $K_{4,n}$

$$K_{4,n}^{(m)} = \frac{7}{192}(K_{4,n+1} - K_{4,n-1}) + \frac{1}{32}(K_{4,n+2} - K_{4,n-2}) - \frac{1}{192}(K_{4,n+3} - K_{4,n-3}).$$

The normalization condition $K_{4,n}^{(m)} + K_{4,n}^{(n)} = \frac{1}{2}K_{4,n}$, which is satisfied for all m, allows us to find the limit values of the occupation probabilities: $R_4^{(m)} = 1/(2\sqrt{3})$, $R_4^{(n)} = 1/2 - 1/(2\sqrt{3})$. Closed form expressions for the probabilities $R_m^{(m)}$ and $R_m^{(n)}$, containing radicals, can be obtained for some other values of m. For example, for $m = 5$, we have

$$R_5^{(m)} = \frac{\sqrt{30 + 2\sqrt{205}}}{5\sqrt{41}}, \quad R_5^{(n)} = \frac{1}{2} - \frac{\sqrt{30 + 2\sqrt{205}}}{5\sqrt{41}}.$$

In our work, the inverse power dependence was used to describe the transition to the thermodynamic limit. Such asymptotic behavior is quite common for many lattice models. Fitting the values of $R_m^{(m)}$ and $R_m^{(n)}$ obtained at $2 < m < 12$ in accordance with the function $A + B/m^{\alpha}$, we obtain the following results:

$$R_m^{(m)} \sim 0.2498 - \frac{0.3284}{m^{2.132}}, \quad R_m^{(n)} \sim 0.2502 + \frac{0.3284}{m^{2.132}} \text{ if } m \text{ is odd},$$

$$R_m^{(m)} \sim 0.2489 + \frac{0.6017}{m^{1.959}}, \quad R_m^{(n)} \sim 0.2511 - \frac{0.6017}{m^{1.959}} \text{ if } m \text{ is even}.$$

The parameters A, B, and α were obtained by the least squares method using the `LSSolve` procedure from the `Optimization` package. The scatter of the parameters found allows us to estimate the accuracy of the asymptotic expansion coefficients.

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Confined stochastic disturbance for functional iterations based on a continuous-event model of a biosystem

Perevaryukha Andrey

Abstract. In our report we consider a new method for including in the computer model a special trigger function, which introduces a random value in the right side of the hybrid system of differential equations, with changes in the coefficients in the equation. Stochastic perturbation receives only an iterative trajectory in a limited range in the vicinity of an unstable repeller point. The method uses a continuous-discrete structure of calculations with a set of predicates. The approach makes it possible to simulate the scenario of the collapse of fish stocks with a transition from a stable state to intermediate and limited stochastic fluctuations.

Introduction

Natural processes in the environment have a significant stochastic component. Turbulent effects introduce noise when building hydrological models from observations of the spread of pollution spots [1]. In population dynamics, random factors determine the success of the reproductive cycle for many marine species. The early stages of the life of marine fishes are subject to changeable aquatic environment and the whirlwind of the sea winds. Stochastic continuous models are often used for modeling in biology. However, the stochastic external influence may increase or decrease depending on the state of the biosystem itself. For populations in a normal state, random fluctuations in survival are not visible, they are inferior to the direct action of trophic competitors. Principles change when excessive fish stocks are used up. The purpose of the work is to describe the transition to a collapse when a population falls into the zone of random fluctuations. For several important reasons, the stochastic component $\Theta(N(0))$ must be introduced into a narrow

critical state of a model object. In the report, we will show this fact using the example of $\{\psi(x_0)\}_1^n \to \infty$ iterations, which we obtain by numerically solving the hybrid system of differential equations for the problem of modeling fish collapse.

The novelty of our approach is that we can choose the neighborhood in which the trajectory $x_{n+1} = \psi(x_n, x_{n-1}) - Q_n x_n$, where the value Q_n reflects the impact of exploitation, in computer calculations will show stochastic properties.

1. Hybrid model

Let us further consider the three-stage model of the survival of the generation from $N(0)$. Factors in the loss of abundance will change significantly as the stages of life change, but depending on the density. A complex functional dependence will create thresholds that lead to outbreaks for insects [2] outbreak scenario after accidental release of population state from the control factors control interval Ω_s. For fish situation lead to a rapid collapse of stocks during overfishing.

Survival of generations $R = N(T)$ from $N(0) = \lambda S$,$S \in \Omega_S$ on the interval $t \in [0, \dots \xi, \omega \dots, T]$ we describe the stages of ontogeny by a predictively redefined system:

$$\frac{dN}{dt} = \begin{cases} -(\alpha \bar{w}(\xi) N(t) + \bar{\Theta}(N(0))\beta)\, N(t), 0 < t < \xi \\ -(\alpha_1 N(\xi)/w(\omega) + \beta) N(t), \xi < t < \omega, \\ -(\alpha_2 N(t)) N(t - \varsigma), \omega < t < T \end{cases} \tag{1}$$

$[0, \xi], [\xi, \omega]$ — duration of stages. α, β — indicators of mortality rates. $\Theta(N(0)) = [1 + \exp\left(-\kappa N(0)^2\right)], \lim_{N(0)\to\infty} \Theta(N(0)) = 1$ function determines the threshold reduction in reproduction efficiency for $S < \mathcal{L}$. Let the region of a small group of individuals $\mathcal{L} \subset U_1 \in \Omega_S$, where reproduction of fish is due to random factors that we take into account, complementing the discrete–continuous model by indirect interaction. We will connect $\bar{\Theta}(N(0), w) = \Theta(N(0)) \times w(t)$ with the index of calculating the conditional dimensional development from the second equation: $\dot{w}(t) = [G/(N^{2/3} + \sigma)] \times \gamma$,$w(0) = w_0$, γ — uniformly distributed random variable. Obtained on the basis of unimodal dependence $\psi(x) = \bigcup_{N(0)} N(T), N(0) \in \mathbb{Z}^+$ numerical solutions of the Cauchy problem (1) on the interval $t \in [0, T]$ iteration trajectory $x_{n+1} = \psi(x_n), x_0 < \mathcal{L}$ has the properties of a bounded stochastic perturbation. Instead of a threshold point: $\psi(x_*) = x_* < \max \psi(x)$,$\forall x < x_* - \epsilon$:$\lim_{n\to\infty} \psi^n(x_*) = U_0, U_0 < \epsilon$ so we create an interval of probabilistic behavior of a trajectory that admits an event: $x_0 < x_*, \psi^k(x_0) > \max \psi(x)$ simulating a outbreak situation from a small group — the trajectory comes to a stable regime $\psi^p(x_i) = \psi^{p+2}(x_i), x_i > \max \psi(x)$, where there is no stochasticity.

The model (1) combines stochastic and deterministic behavior in ranges that do not have a smooth boundary for allowable fish stock values Ω_S.

Chaotic fluctuation for fish population in crisis we can obtain in equation with delay:

$$\frac{dN}{dt} = rN(t)\frac{\mu\nu N(t-\tau)}{\nu e^{\nu\tau} + \kappa(e^{\nu\tau}-1)N(t-\tau)} - \kappa N^2(t).$$

Conclusion

Thus, we have obtained a model where the collapse is determined by falling into the regime of a stochastic state with an insignificant number of fish stocks. If the threshold state is represented by a point-repeller, then a speedy state of death of $N(\hat{t}) = 0$ will occur. The complete death of the fish species in the environment is a long-lasting phenomenon [3]. Biological systems are adaptive [4] and fishes able to change time lags in ontogenesis. Fish species shorten the life cycle for survival. A stochastic model may demonstrate an unexpected recovery of a population that has previously experienced a collapse. The described collapse model with recovery was realized for whitefish in Lake Ontario.

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Differentiation and special functions on finite fields

N. V. Proskurin

Аннотация. We consider complex functions over finite fields. In this context, we find finite analogues for some classical special functions. That are the error function and the incomplete gamma function. Our approach is based on consideration of differential equations satisfied by the special functions.

1. Preliminaries

Given prime p, let $\mathbb{F}_q$ be the finite field with $q = p^l$ elements and with prime subfield $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let $e_q\colon \mathbb{F}_q \to \mathbb{C}^\star$ be a non-trivial additive character. With some $h \in \mathbb{F}_q^\star$, one has $e_q(x) = \exp\bigl(2\pi i\, tr(hx)/p\bigr)$ for all $x \in \mathbb{F}_q$ (tr denotes the trace function $\mathbb{F}_q \to \mathbb{F}_p$). We write $\mathbb{F}_q^\star$ for the multiplicative group of $\mathbb{F}_q$, and $\widehat{\mathbb{F}}_q^\star$ for the group of multiplicative characters of $\mathbb{F}_q$, i. e. for the group of homomorphisms $\chi\colon \mathbb{F}_q^\star \to \mathbb{C}^\star$. By ϵ we mean the trivial character, $\epsilon(x) = 1$ for all $x \in \mathbb{F}_q^\star$. Extend each $\chi \in \widehat{\mathbb{F}}_q^\star$ to all of $\mathbb{F}_q$ by setting $\chi(0) = 0$.
The classical trigonometric sums

$$G(\chi) = \sum_{x\in\mathbb{F}_q^\star} \chi(x) e_q(x) \quad \text{with} \quad \chi \in \widehat{\mathbb{F}}_q^\star \qquad \text{(Gauss)}$$

$$J(\alpha,\beta) = \sum_{x\in\mathbb{F}_q} \alpha(x)\beta(1-x) \quad \text{with} \quad \alpha,\beta \in \widehat{\mathbb{F}}_q^\star \qquad \text{(Jacobi)}$$

may be considered as finite analogues of Euler's gamma and beta functions.

2. Elements of analysis

Consider the complex vector space Ω_q of all functions $\mathbb{F}_q \to \mathbb{C}$. Given character $\eta \in \widehat{\mathbb{F}}_q^\star$, define the linear operator $D^\eta\colon \Omega_q \to \Omega_q$ by

$$D^\eta F(x) = \frac{1}{G(\bar\eta)} \sum_{t\in\mathbb{F}_q} F(t)\, \bar\eta(x-t)$$

for all $F\colon \mathbb{F}_q \to \mathbb{C}$ and $x \in \mathbb{F}_q$. We then have

$$D^{\epsilon}F(x) = F(x) - \sum_{t\in\mathbb{F}_q} F(t),$$

$$\frac{1}{G(\eta)} D^{\eta}F(x) = \frac{1}{q} \sum_{t\in\mathbb{F}_q} F(t)\,\bar{\eta}(t-x)$$

for all F and x as above and $\eta \neq \epsilon$. By Evans [1], $D^{\eta}F$ is the derivative of order η of F. This definition is motivated by analogy with the Cauchy integral formula for the derivatives of analytic functions.

One finds easily, D^{η} takes constant functions to zero function, whenever $\eta \neq \epsilon$, and $D^{\alpha}D^{\beta} = D^{\alpha\beta}$ for characters α, β subject to $\alpha\beta \neq \epsilon$. Also, given two functions E and F, $x \in \mathbb{F}_q$, and the character ν we have the formula for integration by parts

$$\sum_{x\in\mathbb{F}_q} E(x)\, D^{\nu}F(x) = \nu(-1) \sum_{x\in\mathbb{F}_q} F(x)\, D^{\nu}E(x)$$

and the Leibniz rule for the ν-th derivative of the product

$$D^{\nu}EF(x) = \frac{1}{q-1} \sum_{\mu\in\widehat{\mathbb{F}}_q^{\star}} \frac{G(\bar{\mu})G(\mu\bar{\nu})}{G(\bar{\nu})}\, D^{\mu}E(x)\, D^{\nu\bar{\mu}}F(x).$$

Clearly, this differentiation differs significantly from the one considered in the classical analysis and in the differential algebra. In the classical setting, the orders of derivatives are integers, i. e. elements of the additive group $\mathbb{Z}$. The orders η of the derivatives D^{η} are elements of the finite cyclic group $\widehat{\mathbb{F}}_q^{\star}$.

Let $F(x) = e_q(-x)$ for all $x \in \mathbb{F}_q$. Given any character χ, we have $D^{\chi}F = F$. More generally, if $c \in \mathbb{F}_q$ and $F(x) = e_q(-cx)$ for all $x \in \mathbb{F}_q$, then $D^{\chi}F(x) = \chi(c)F(x)$ for all $x \in \mathbb{F}_q$. It follows, for any $F \in \Omega_q$ given by the Fourier series

$$F(x) = \sum_{m\in\mathbb{F}_q} C(m) e_q(-mx)$$

one has

$$D^{\chi}F(x) = \sum_{m\in\mathbb{F}_q} \chi(m)\, C(m) e_q(-mx)$$

for all $x \in \mathbb{F}_q$.

3. Error function

The complimentary error function erfc is entire function defined by the probability integral

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2)\, dt \quad \text{for } \ z \in \mathbb{C}.$$

We can treat erfc as the only solution to the differential equation

$$w'(z) = -\frac{2}{\sqrt{\pi}} \exp(-z^2) \quad \text{with } \ w(0) = 1.$$

We are interested in a finite field analogue of erfc. Take any $\nu \in \widehat{\mathbb{F}}_q^\star$ and $c \in \mathbb{F}_q$. Assume $\nu \neq \epsilon$ and $c \neq 0$. As an analogue of the equation above, consider the differential equation

$$D^\nu w(z) = c\, e_q(z^2) \quad \text{for all } \ z \in \mathbb{F}_q.$$

Solving the equation for w we get

$$w(z) = \frac{c}{G(\nu)} \sum_{t \in \mathbb{F}_q} \nu(t)\, e_q\big((z-t)^2\big) + d$$

with arbitrary $d \in \mathbb{F}_q$. We are free in choice of the constants c and d. Let erfc_ν be the function w above with $c = G(\nu)$ and $d = 0$. So that

$$\operatorname{erfc}_\nu(z) = \sum_{x \in \mathbb{F}_q} \nu(z+x)\, e_q\big(x^2\big) \quad \text{for all } \ z \in \mathbb{F}_q.$$

We consider erfc_ν as a finite field analogue of complimentary error function erfc. (It depends on character ν used in the definition of differentiation.)

4. Incomplete gamma function

For the incomplete gamma function one has

$$\Gamma(s+1, z) = \int_z^\infty e^{-x} x^s dx \quad \text{for } \ s \in \mathbb{C} \ \text{ and } \ z \in \mathbb{C} \setminus (-\infty, 0].$$

Fix any $s \in \mathbb{C}$. The function $w = \Gamma(s+1, \cdot)$ is the only solution to the differential equation

$$w'(z) = -e^{-z} z^s \quad \text{with } \ w(0) = \Gamma(s+1).$$

Its finite analogue should be a complex function over $\widehat{\mathbb{F}}_q^\star \times \mathbb{F}_q$. Take any $\nu \in \widehat{\mathbb{F}}_q^\star$. Assume $\nu \neq \epsilon$ and consider the equation

$$D^\nu w(z) = c\, e_q(z)\, \mu(z)$$

with some $\mu \in \widehat{\mathbb{F}}_q^\star$ and $c \in \mathbb{F}_q^\star$. Solving the equation for w we obtain the function $\Gamma_\nu \colon \widehat{\mathbb{F}}_q^\star \times \mathbb{F}_q \to \mathbb{C}$ which we treat as finite analogue of the incomplete gamma function. Certainly, it depends on character ν involved into D^ν. Explicitly,

$$\Gamma_\nu(\mu, x) = \sum_{t \in \mathbb{F}_q} \mu(t)\,\nu(t-x)\,e_q(t) \quad \text{for all } \ \mu \in \widehat{\mathbb{F}}_q^\star \ \text{ and } \ x \in \mathbb{F}_q.$$

For $\mu, \nu \in \widehat{\mathbb{F}}_q^\star$ and $x \in \mathbb{F}_q$, let us introduce the sum

$$E(\mu, \nu; x) = \sum_{u \in \mathbb{F}_q} \mu(u)\,\nu(1-u)\,e_q(xu).$$

The classical Gauss and Jacobi sums are united in the sum E. Indeed, one has $E(\mu, \nu; 0) = J(\mu, \nu)$ and one can evaluate $E(\mu, \nu; x)$ in terms of $G(\mu)$ or $G(\nu)$ if either $\nu = \epsilon$ or $\mu = \epsilon$.
It occures, our function E is the main part of the incomplete gamma function. Precisely, one has

$$\Gamma_\nu(\mu, x) = \mu(x)\,\nu(-x)\,E(\mu, \nu; x) \quad \text{for all } \ x \in \mathbb{F}_q^\star.$$

One has also $\Gamma_\nu(\mu, 0) = G(\mu\nu)$.

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Some new identities for Bernoulli numbers via central Krawtchouk polynomials

Nikita Reporez

Abstract. Some new identities involving Bernoulli numbers are proposed.

In this paper we demonstrate some new identities involving central Krawtchouk polynomials $\dot{\mathcal{K}}_m(z)$ defined as $\dot{\mathcal{K}}_m(z) = K_m^{(2m)}(z) = \sum_{j=0}^{m}(-1)^j \binom{z}{j}\binom{2m-z}{m-j}$, $m \geq 0$, where $K_m^{(N)}(z)$ denotes Krawtchouk polynomial of order N and degree m [1], and the sequence of the Bernoulli numbers $\{\ b_j\}_{j\geq 0}$ with $b_1 = -\frac{1}{2}$ [2] .
We assert that the following below identities (1)-(6) are valid:

1.
$$\sum_{j=0}^{m}(\#_j\dot{\mathcal{K}}_m)(2-2^j)b_j = \frac{4^m}{m+1} \qquad , \qquad m \geq 0 \tag{1}$$

2. Starting from this item we introduce an "umbral variable" [3] ϕ as a following rule for "change of variables":
$$\phi = \{\phi^j\} : \phi^j \Longrightarrow (2-2^j)b_j \qquad , \qquad j \geq 0\ .$$
So, any polynomial on variable ϕ, expanded by powers of ϕ, after applying to it the shown above "change of variables", becomes a linear combination of the Bernoulli numbers.
In these notations identity (1) can be written in "umbral form" as follows:
$$\dot{\mathcal{K}}_m(\phi) = \frac{4^m}{m+1} \qquad , \qquad m \geq 0 \tag{2}$$

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3. From the properties of the central Krawtchouk polynomials proved in [4, Lemma 3] and identity (2) one easily gets identity :

$$\binom{\frac{\phi-1}{2}}{m} = \frac{(-1)^m}{m+1} \qquad , \qquad m \geq 0 \tag{3}$$

where binomial coefficient

$$\binom{\frac{\phi-1}{2}}{m}$$

is being considered as a polynomial on ϕ with the consequential change $\phi^j \Longrightarrow (2-2^j)b_j$.

For example, if $m = 4$ the expansion of the binomial coeffifcient $\binom{\frac{\phi-1}{2}}{4}$ is $\frac{35}{128} - \frac{11\phi}{24} + \frac{43\phi^2}{192} - \frac{\phi^3}{24} + \frac{\phi^4}{384}$, and changing $\phi^j \Longrightarrow (2-2^j)b_j$ for $0 \leq j \leq 4$, one gets $\frac{35}{128} \cdot (2-2^0) \cdot 1 - \frac{11}{24} \cdot (2-2^1) \cdot (-\frac{1}{2}) + \frac{43}{192} \cdot (2-2^2) \cdot (\frac{1}{6}) - \frac{1}{24} \cdot (2-2^3) \cdot 0 + \frac{1}{384} \cdot (2-2^4) \cdot (-\frac{1}{30}) = \frac{1}{5}$ in full accordance with (3).

4. From (3) easily follows that

$$\sum_{m=0}^{n-1} (-1)^m \binom{\frac{\phi-1}{2}}{m} = H_n \qquad , \qquad n \geq 1, \tag{4}$$

where H_n is the n-th Harmonic number [2].

5. At last, again from formula (3) we obtain an equality for the ordinary generating functions of its left and right sides:

$$(1+t)^{\frac{\phi-1}{2}} = \frac{\ln(1+t)}{t} \qquad , \qquad |t| < 1. \tag{5}$$

In particular for $t = -\frac{1}{2}$ in (5) we get a nice "umbral" equality

$$\Big(\frac{1}{2}\Big)^{\frac{\phi-1}{2}} = \sum_{k=0}^{\infty} \Big(-\frac{1}{2}\Big)^k \binom{\frac{\phi-1}{2}}{k}\Big|_{\phi^j \to (2-2^j)b_j} = 2\ln 2. \tag{6}$$

Remark.
All the above equalities (1)-(6) were approved with MATHEMATICA-10.

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On polynomials of odd degree over reals

Alexandr V. Seliverstov

Abstract. We prove that there exists a large set of real hypersurfaces having elliptic points. In particular, for almost all nonlinear polynomials of odd degree in two or three variables, the graph of the polynomial contains an elliptic point. Some polynomials in many variables are also considered.

Let us denote by $\mathcal{F}$ an affine hypersurface, that is, the vanishing locus of a polynomial over the field of real numbers. A smooth point $P \in \mathcal{F}$ is said to be elliptic if it is the isolated real point of the intersection of the hypersurface $\mathcal{F}$ with the tangent hyperplane $\mathcal{T}_P$ to $\mathcal{F}$ at this point and the second fundamental form is positive definite. Roughly speaking, in a sufficiently small analytic neighborhood of an elliptic point, the hypersurface looks like an ellipsoid. If the affine hypersurface is a graph of a polynomial f, then the matrix of second partial derivatives $\frac{\partial^2 f}{\partial x_j \partial x_k}$ is definite at every elliptic point.

In a sufficiently small analytic neighborhood of an elliptic point, all points of the hypersurface are elliptic. If a polynomial is defined over the field of rational numbers, then its graph contains an everywhere dense set of rational points. Consequently, if there exists an elliptic point on the graph, then there exists a rational elliptic point. It is easy to check whether a given rational point is elliptic. In practice, one can use software for symbolic computations [1].

The term "almost all" means "all but a set covered by a vanishing locus of a nonzero polynomial".

Theorem 1. *Given an odd integer $d \geq 3$. For almost all bivariate polynomials of degree d, the graph of the polynomial contains an elliptic point.*

Remark. In accordance with Theorem 1, for almost all nonlinear bivariate polynomials of odd degree, there exists a point, where the matrix of second partial derivatives is definite. Thus, for each bivariate polynomial of odd degree, there exists a point, where this matrix is semidefinite. In particular, for linear polynomials, the matrix vanishes.

Example. The graph of the polynomial $x_1(x_1^2 - 3x_2^2)$ is the monkey saddle; it has no elliptic point. But there exists a tangent plane that intersect the surface

along three straight lines meeting at one point. At the origin all second partial derivatives of the polynomial vanish.

Theorem 2. *Given an odd integer $d \geq 3$. For almost all polynomials of degree d in three variables, the graph of the polynomial contains an elliptic point.*

Remark. On the general three-dimensional cubic hypersurface over the field of complex numbers, straight lines fill the whole hypersurface. If this property holds over the field of real numbers, then the hypersurface has no elliptic point. Nevertheless, there is a large set of real cubic hypersurfaces having an elliptic point.

Theorem 3. *Given both integer $n \geq 1$ and odd integer $d \geq 3$. For almost all $(n+1)$-tuples of linear functions in n variables, the graph of the sum of d-th powers of the linear functions contains an elliptic point.*

Remark. In the general case, a linear function is inhomogeneous.

Example. Let us consider a cubic form of the type $x_1^3 + x_2^3 + \mu(x_1 + x_2)^3$. The matrix of second partial derivatives is equal to

$$6 \begin{pmatrix} (1+\mu)x_1 + \mu x_2 & \mu(x_1 + x_2) \\ \mu(x_1 + x_2) & \mu x_1 + (1+\mu)x_2 \end{pmatrix}.$$

Its determinant is equal to $36\left(\mu x_1^2 + \mu x_2^2 + (1+2\mu)x_1 x_2\right)$. If $x_1 = x_2 > 0$ and $\mu > -1/4$, then the matrix is positive definite. On the other hand, if $\mu = -1$, then the determinant is equal to $-36(x_1^2 + x_1 x_2 + x_2^2) \leq 0$. In this case, for all points, the matrix is neither positive nor negative definite.

Theorems 1–3 can be used to develop new heuristic or generic algorithms (cf. [2]) to check some properties of real hypersurfaces. On the other hand, the recognition of elliptic points on a surface can be used in computer-aided geometric design.

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Apparent singularities of D-finite systems

Shaoshi Chen, Manuel Kauers, Ziming Li and Yi Zhang

Abstract. We generalize the notions of ordinary points and singularities from linear ordinary differential equations to D-finite systems. Ordinary points and apparent singularities of a D-finite system are characterized in terms of its formal power series solutions. We also show that apparent singularities can be removed like in the univariate case by adding suitable additional solutions to the system at hand. Several algorithms are presented for removing and detecting apparent singularities. In addition, an algorithm is given for computing formal power series solutions of a D-finite system at apparent singularities.

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First order linear ODE equivalent to deformed Heun equation

S. Slavyanov and O. Stesik

The Heun equation is a Fuchsian equation with four Fuchsian singularities $z_1 = 0$, $z_2 = 1$, $z_3 = t$, $z_4 = \infty$. It can be written as

$$(H(d/dz, z) - h)w(z) = 0, \text{ with}$$

$$H(d/dz, z) = \frac{1}{f(t)}[\sigma(z)\frac{d}{dz}^2 + \tau(z)\frac{d}{dz} + \omega(z)],\ f(t) = t(t-1),$$

$$\sigma(z) = \prod_{z=1}^{3} z - z_j,\ \tau(z) = \sum_{j=1}^{3}(1-\Theta_j)\frac{\sigma(z)}{z-z_j},\ \omega(z) = \alpha(2 - \sum_{j=1}^{3}\Theta_j - \alpha)z \qquad (1)$$

The deformed Heun equation arises at addition one apparent singularity located at $z_5 = q$ into equation (1) with the requirement that the derivative of those solution normalized as $w(q) = 1$ is equal to $w'(q) = \mu$.

It can be presented in the symmetric operator form as

$$(H(p, z) - H(\mu, q))w(z) + \frac{\mu\sigma(q)w(z) - \sigma(z)w'(z)}{f(t)(z-q)} = 0 \qquad (2)$$

Heun equation (1) and deformed Heun equation (2) both can generate Painlevé equation P^6 with the help of antiquantization procedure [1], [2], [3], [4], [5], [6].

Let the 2×2 linear system corresponding to deformed Heun equation (2) be

$$\vec{Y}'(z) = A(z)\vec{Y}(z) = \left(\sum_{j=1}^{3}\frac{A^{(j)}}{z-z_j}\right)\vec{Y}(z), \qquad (3)$$

where $\vec{Y}(z)$ is a vector with components $y_1(z), y_2(z)$ and $A^{(j)}$ are residue matrices at singularities with constant matrix elements $a_{ik}^{(j)}$ and zero determinants $\det A^{(j)} = 0$. The eigenvalues of $A^{(j)}$ are denoted by 0 and Θ_j; then the following equalities between matrix elements are valid as a result of the condition on the determinants

$$a_{22}^{(j)} = \Theta_j - a_{11}^{(j)}, \quad a_{21}^{(j)}a_{12}^{(j)} = a_{11}^{(j)}(\Theta_j - a_{11}^{(j)}). \qquad (4)$$

Further we assume that the matrix-residue at infinity is diagonal

$$A^{(\infty)} = \begin{pmatrix} \sum_{j=1}^{3} a_{11}^{(j)} & 0 \\ 0 & \sum_{j=1}^{3}(\Theta_j - a_{11}^{(j)}) \end{pmatrix} \tag{5}$$

This condition leads to two equations binding the off-diagonal matrix elements of matrix residues at finite singularities z_j.

$$\sum_{j=1}^{3} a_{12}^{(j)} = 0 \sum_{j=1}^{3} a_{21}^{(j)} = 0 \tag{6}$$

Solving equations (6) it is possible to exclude $a_{21}^{(3)}$ and $a_{12}^{(3)}$ from further computations.

Hence. the off-diagonal matrix elements a_{12} and a_{21} are linear function of z and can be presented as

$$\begin{aligned} a_{12} &= -(a_{12}^{(1)}t + a_{12}^{(2)}(t-1))z + a_{12}^{(1)}t = \mu_1(z-q_1) \\ a_{21} &= -(a_{21}^{(1)}t + a_{21}^{(2)}(t-1))z + a_{21}^{(1)}t = \mu_2(z-q_2) \end{aligned} \tag{7}$$

Presentation (7) gives four equations for μ_i, q_i as functions of $a_{ik}^{(j)}$ and vice versa. Namely

$$a_{12}^{(1)} = \frac{-\mu_1 q_1}{t},\ a_{12}^{(2)} = \frac{\mu_1(q_1-1)}{t-1},\ a_{21}^{(1)} = \frac{-\mu_2 q_2}{t},\ a_{21}^{(2)} = \frac{\mu_2(q_2-1)}{t-1} \tag{8}$$

Turning to (6) it holds

$$a_{11}^{(1)} = \frac{\Theta_1}{2} \pm \sqrt{\frac{\Theta_1^2}{4} + \frac{\mu_1^2 q_1^2}{t^2}},\quad a_{11}^{(2)} = \frac{\Theta_2}{2} \pm \sqrt{\frac{\Theta_2^2}{4} + \frac{\mu_2^2(q_2-1)^2}{(t-1)^2}} \tag{9}$$

Combining all equations (4), (6), (7) we appear at nine equations binding nine matrix elements $a_{ik}^{(j)}$ and classical parameters μ_i, q_1 as a consequence.

Second order equations for the functions $y_k(z)$, $r = 1, 2$ would be

$$y_1(z)'' + P_1(z)y_1(z)' + Q_1(z)y_1(z) = 0,\quad y_2(z)'' + P_2(z)y_2(z)' + Q_2(z)y_2(z) = 0, \tag{10}$$

where

$$P_1(z) = -\log' a_{12} - \mathrm{tr}A,\quad Q_1(z) = \det A - a_{12}\left(\frac{a_{11}}{a_{12}}\right)',$$

$$P_2(z) = -\log' a_{21} - \mathrm{tr}A,\quad Q_2(z) = \det A - a_{21}\left(\frac{a_{11}}{a_{21}}\right)'$$

and the prime means the derivative with respect to z. Rather boring computations for which the use of any ACS is preferred show that equations (10) appear to be a deformed Heun equations. At the final step Painlevé equation P^6 can be generated.

Our study is alternative to that in [7] where the linear 2×2 first order system equivalent to Heun equation was constructed. It should be mentioned that for the latter the residue matrix at infinity is not diagonal but triangular.

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BellY polynomials in Mathematica and asymptotic solutions of integral equations.

O. Marichev and S. Slavyanov

The importance of classical polynomials is widely recognized in the theory of ordinary linear differential equations. Less known is the role of Bell polynomials in the theory of functional equations both linear and nonlinear. Here a short review of modern state of BellY polynomials implemented in CAS Mathematica is given and also an example of their use at solution of functional equations arising in the theory of integral equations with rapidly oscillating kernels is exposed.

Asymptotic solution of Fredholm equation with a large parameter in the exponent leads to the necessity for solution the following two functional equations [1], [2] one linear and the other nonlinear

$$\frac{\partial F(z,s)}{\partial s} + \frac{\partial F(s,t)}{\partial s} = 0 \tag{1}$$

with $s = \phi(z)$, $t = \phi(\phi(z))$.

$$h(z) = \mu h(\phi(z)) T(z, \phi(z)) \tag{2}$$

Here $\phi(z)$ and $h(z)$ are unknown functions denoted as the phase and the amplitude while $F(z,\phi)$ and $T(z.\phi)$ are known functions which are regular in an appropriate vicinity of the origin. The parameter μ plays the role of an eigenvalue.

The rigorous mathematical approach to the posed problem is extremely difficult since the small denominators arise. However abstracting from these mathematical heights the approximate solution in terms of polynomials can be constructed near the origin. The actions with polynomials include differentiation of superposition of functions and calculation the inversion of functions. For these purposes handling with BellY polynomials implemented in CAS Mathematica can give a needed help. The proposed approach can be used in various physical problems, for instance, in the theory of open resonators.

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The problem of resolution of singularities in arbitrary characteristic.

Mark Spivakovsky

The subject of this talk is the problem of resolution of singularities in algebraic geometry, but it is intended for a more general audience of algebraists than just specialists in birational geometry. The problem of resolution of singularities asks whether, given an algebraic variety X over a field, there exists a non-singular algebraic variety X' and a proper map $X' \to X$ which is one-to-one over the non-singular locus of X. If we cover X' by affine charts, the problem becomes one of *parametrizing* pieces of X by small pieces of the Euclidean space k^n. This localized version of the problem, called Local Uniformization, is usually stated in terms of valuations and can be interpreted as follows. Let (R, M, k) be a local quasi-excellent noetherian domain (resp. a local k-algebra essentially of finite type without zero divisors) and let R_ν be a valuation ring containing R and having the same field of fractions as R. Find a *smooth* finite type R-algebra R' such that $R' \subset R_\nu$. The **Local Uniformization Theorem** asserts the existence of such an R'; it was proved by O. Zariski in 1940 in the case when $char\ k = 0$ and is one of the central open problems in the field when $char\ k = p > 0$.

To study local uniformization we will introduce the notion of key polynomials associated to a simple extension $\iota : K \to K(x)$ of valued fiels, defined by Saunders Mac Lane in the 1930-ies in the case of discrete valuations and generalized by M. Vaquié, F.J. Herrera Govantes, J. Decaups, W. Mahboub, M. A. Olalla Acosta, J. Novacoski and M. Spivakovsky.

We will discuss the applications of key polynomials to the problem of Local uniformization in arbitrary characteristic.

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Numerical Symbolic Dynamics: Studies of the Invariant Components

Chase Tisagh, Mylläri Aleksandr, Mylläri Tatiana and Vassiliev Nikolay

Abstract. We consider equal mass free-fall three-body problem. We construct numerically symbolic sequences using close binary approaches and analyze components revealed on the scatterplot of maximum values of the entropy - corresponding length of symbolic sequence and as peaks on the histogram for the Shannon entropy of these sequences.

Introduction

Symbolic dynamics was used to analyze some special cases of the three-body problem: Alexeyev [2, 3, 4, 5] has found theoretically an intermittence of motions of different types in the one special case of the three-body problem - Sitnikov problem. Symbolic dynamics was also applied in two other special cases of the three-body problem: the rectilinear problem (numerically) (Tanikawa & Mikkola [10, 11]); and the isosceles problem (Zare & Chesley [12, 6]).

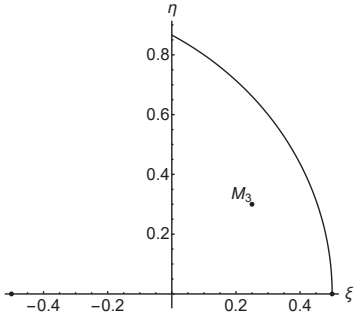

FIGURE 1. Agekian-Anosova region D.

Equal mass free-fall three-body problem is convenient for study since it allows easy visualization of initial configuration: if we place two bodies in the points

$(-0.5; 0)$ and $(0.5; 0)$, then all possible configurations will be covered if we place the third body inside the region D bounded by two straight line segments and arc of the unit circle centered at $(-0.5, 0)$ (Fig. 1) [1]. Here, we analyze components revealed on the scatterplot of maximum values of the entropy - corresponding length of symbolic sequence and as peaks on the histogram for the maximum Shannon entropy of the symbolic (sub-)sequences constructed using close binary approaches that were found earlier [9]. We used symplectic code by Seppo Mikkola (Tuorla Observatory, University of Turku) [8] for numerical simulations.

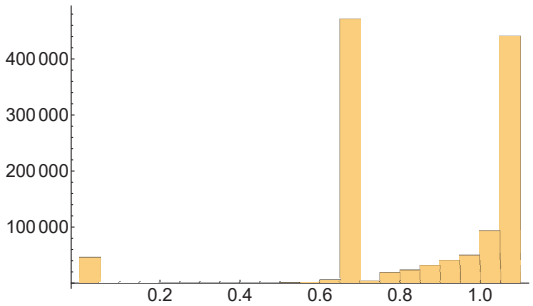

FIGURE 2. Histogram of the maximum values of Shannon entropy for sub-sequences.

Analysis of structures

We integrate equations of motion numerically and construct symbolic sequences during the process using close binary approaches: we detect minimum distance between two bodies, and corresponding symbol is the number of the distant body. Thus, our symbols are from the alphabet {1, 2, 3}. Some systems disrupt fast, so some sequences are short. In this case end of the sequence is padded by $0s$. Some systems live long (e.g. metastable systems [7]), so corresponding sequences are long. To have a reasonable computing time, we constructed symbolic sequences of length 100. Since we are interested in the analysis of active three-body interactions, we consider sub-sequences of each of these sequences, increasing the length step-by-step, calculate Shannon entropy for each of these sub-sequences, and find maximum value of these entropies. Maximum value (and moment of time/length of the sub-sequence) correspond to the stage of active interaction between bodies.

Histogram of the maximum values of Shannon entropy for sub-sequences has three distinct modes corresponding to 3 different types of sequences: sequences consisting of only one symbol, sequences consisting of two symbols equally represented, and sequences with all three symbols equally present (Fig. 2). Some structures can also be seen on the scatterplot of maximum values of the entropy - corresponding length of symbolic sequence in the neighborhood of these modes

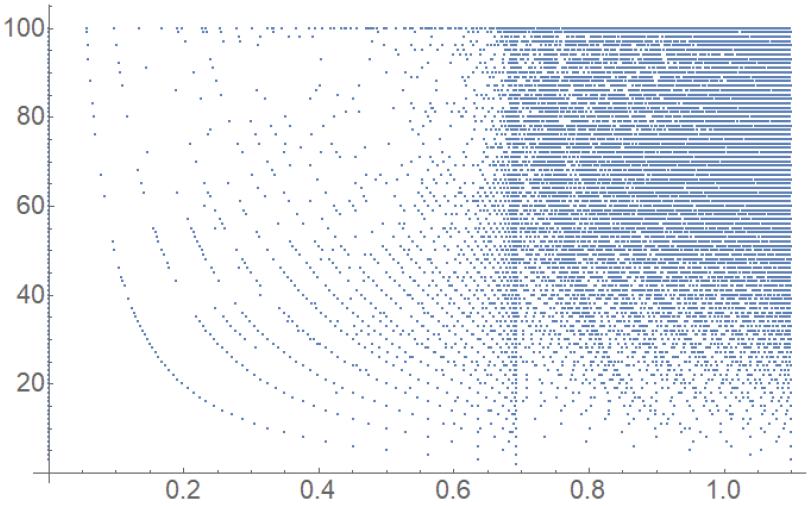

FIGURE 3. Scatterplot of maximum values of the entropy - corresponding length of symbolic sequence.

(Fig. 3). Here, we analyze these structures. In particular, we trace initial conditions and structure of the sequences corresponding to the lines in the left part of Fig.3. Just one point on Fig.3 can correspond to many sequences, see e.g. Fig.4.

Figures 5 and 6 show two feachers of Fig.3 and corresponding initial conditions. Points forming these curves correspond to the sequences of decreasing length that are constructed from two symbols only: one (leftmost curve) or two (second curve) symbols of one kind, and all other symbols of another kind. We trace the structure of these sequences and corresponding initial conditions.

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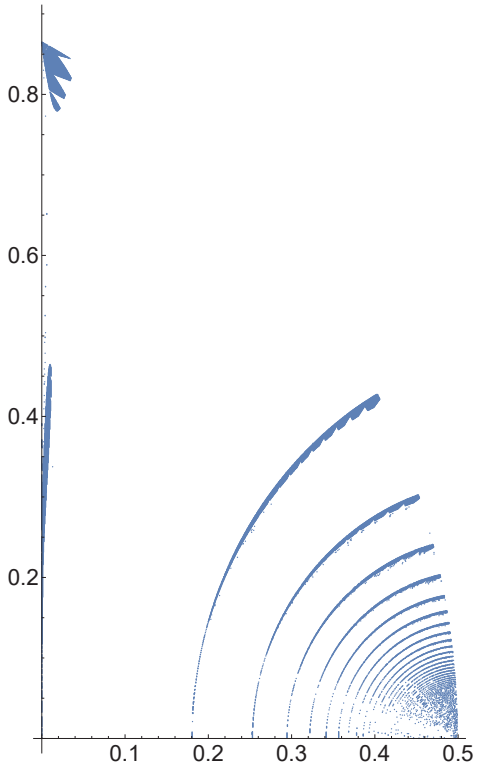

FIGURE 4. Initial conditions corresponding to the point (0.,100) in upper-left corner on Fig.3 - 29321 points.

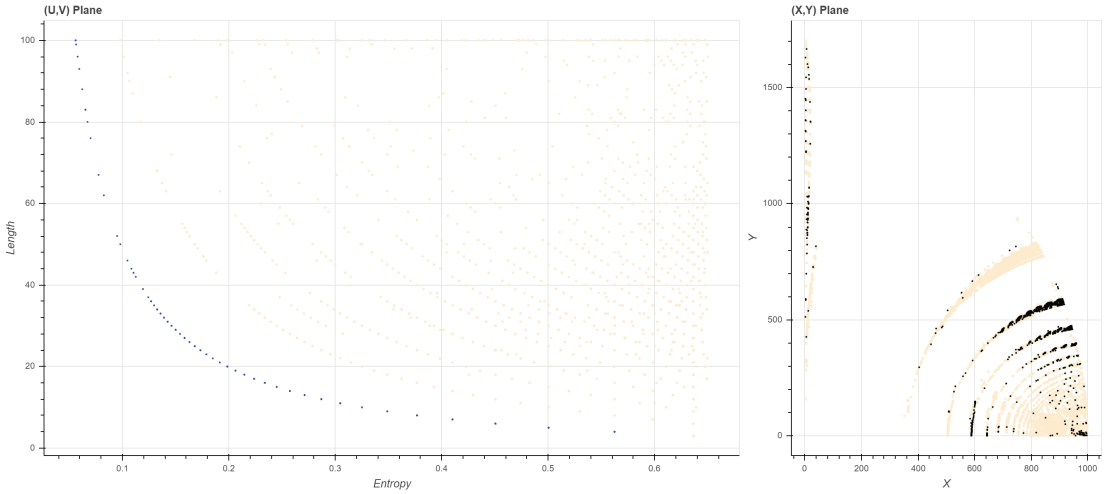

FIGURE 5. Left: leftmost curve on Fig. 3 selected. Right: corresponding initial conditions.

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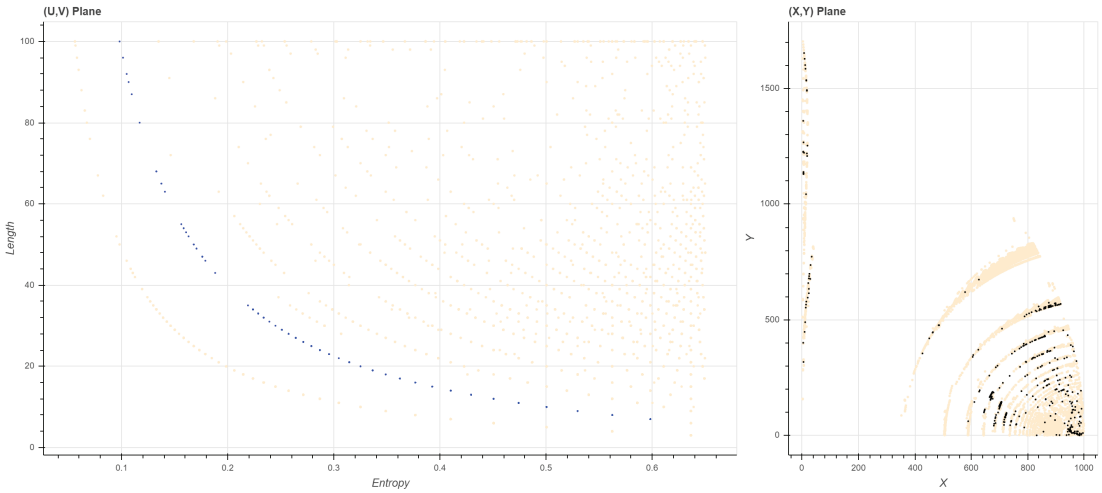

FIGURE 6. Same as Fig. 5, second curve selected.

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On computer algebra aided generation of exact solutions for Fredholm integro-differential equations

K.D. Tsilika

Abstract. We introduce a topic in the intersection of symbolic mathematics and computation, in the fields of Boundary Value Problems for linear integral equations. Our computational approach gives emphasis to mathematical methodology and aims at both symbolic and graphical results. It is implemented in a widely used computer algebra system, *Mathematica*. We develop a solver for unique solutions of Fredholm integro-differential equations in a Mathematica notebook, that displays analytical formulations that can be called up directly. Our easy-to-use program provides in one entry, exact solutions for the abstract operator equation

$$Bu = Au - gF(Au) = f, D(B) = \{u \in D(A) : \Phi(u) = N\Psi(u)\}, u \in D(A), f \in Y \quad (1)$$

Our routine could make a research tool for a wide range of scientists, as BVP for integro-differential equations are often at the forefront of mechanics, physics, biosciences and finance. The code is written in *Mathematica* (v. 11.3). As the interpretation of the code is immediate, it allows ample space for improvements and customization.

Introduction

Applied sciences study phenomena mathematically formulated as Fredholm integro-differential equations subject to boundary conditions. In this study we use the symbolic computation program *Mathematica* [7] in order to generate the symbolic and graphical representation of the exact solutions of Fredholm integro-differential equations assuring their existence first. Our computational approach does not require solving the integro-differential equation with built-in functions and makes no assumptions for the initial conditions. The theoretical methodology comes from the work in [1, 2, 3, 4].

Analytical mathematical methods due to their complexity are not generally clear and immediately comprehensible to a large pool of scientists. In addition,

computations work out faster with computer software, even faster when creating automatically the existence conditions and the intermediate results required before the analytic solution. The computer codes are fully presented and can be reproduced as they are in computational-based research practice. Results of solution steps obtained as outputs are created in a way as to be interpreted without the knowledge of the theoretical methodology. Mathematica has a dedicated function to symbolically solve an integro-differential equation (solves also Fredholm integral equations), DSolveValue (new feature in *Mathematica* v. 11). However, this function seems applicable only for low input parameters and does not give results in most models. Built-in functions NDSolve, NDSolveValue fail also. Numerical solutions of integro-differential equations (using numerous approximations e.g. Laplace transform methods), is a topic of interest in *Mathematica* fora (`https://mathematica.stackexchange.com/questions/24626`).

The paper is organized as follows: after a brief introduction into the mathematical context in section 1, we explain the workings of the code in section 2. Guidance on how to use and/or change the commands and adapt it to other cases is provided.

1. Mathematics Background

Consider the boundary value problem of the type

$$Bu = Au - gF(Au) = f, u \in D(A), f \in Y, \tag{2}$$

$$D(B) = \{u \in D(A) : \Phi(u) = N\Psi(u)\}, \tag{3}$$

$A : X \to X$ is an ordinary m order differential operator

$$Au(x) = \alpha_0 u^{(m)}(x) + \alpha_1 u^{(m-1)}(x) + ... + \alpha_m u(x), \quad \alpha_i \in \mathbb{R}, D(A) = X_A^m$$

where $X = C[a,b]$ or $X = L_p(a,b)$, $p \geq 1$, $X_A^m = C^m[a,b]$, if $X = C[a,b]$, or $X_A^m = W_p^m(a,b)$, if $X = L_p(a,b)$.

$F(Au) = col(F_1(Au), \ldots, F_n(Au))$ is a functional vector representing the integral part of the integro-differential equation, $g = (g_1, \ldots, g_n)$ is a vector of X^n, N a constant $m \times l$ matrix, $\Phi = col(\Phi_1, \ldots, \Phi_m)$, $\Psi = col(\Psi_1, \ldots, \Psi_l)$, are functional vectors with $\Psi(u)$ standing for the multipoint or integral part of the boundary conditions. Let $z = (z_1, z_2, \ldots, z_m)$ be a basis of $kerA$.

Boundary value problems $B : X \to X$ of the type of (1) for the specific case of $l = n$ have been studied by Vasiliev, Parasidis, Providas in [4], using the extension method. The extension method is a generalization of the direct method, which is presented in [4]. Here, problem (1) is investigated and solved also for the case $l \neq n$, $X \neq Y$. We assume multipoint or nonlocal integral boundary conditions, which allows us to consider a very large class of problems for the equation (1). The ultimate result is the exact solution of problem (1).

2. Program Explanation

In this section we propose how *Mathematica* resources can display analytic solutions of Fredholm IDEs that can be called up directly. All formulations come from simple code, with symbolic computations, matrix-vector multiplication, products of operators, as defined by the solution methods proposed in [4]. The subroutine solely uses simple *Mathematica* 's built-in functions as Inverse and Det.

The user must set the input parameters:

The parameters in problem (1)

m = the order m of the differential operator A,
l = the number of components or the dimension l of the functional vector Ψ,
n = the dimension n of the functional vector F,

The structural elements of (2)

F = functional vector F is the integral part of the IDE
g(.)=the functional vector on the left hand side of (2)
f(.)=the function on the right hand side of (2)
t= the list with the values of the variable in the boundary conditions
z= the vector with the basis z of $KerA$ biorthogonal to Φ

The structural elements of the boundary conditions

nmatrix = the $m \times l$ matrix N satisfying the matrix equation $\Phi(u) = N\Psi(u)$
Ψ(.) =the functional vector Ψ of the matrix equation $\Phi(u) = N\Psi(u)$

Par example, to define a particular F and Ψ with n=2, l=2 write
$F[function\_] := \{\int_0^1 x^2 * functiondx, \int_0^1 functiondx\}$
$\Psi[function\_] := \{function/.t \to ti[[1]], function/.t \to ti[[2]]\}$

The output consists of the following results:
1. W,V=the matrices in the condition for the injectivity of B (existence condition also)
2. Det[V], Det[W]=Determinants of W,V needed for the necessary and sufficient condition of injectivity of B
3. Automated testing for injectivity of B or the existence criterion as defined in [4]
4. solution= the analytic solution of Fredholm integro-differential equation
5. Plot the solution in the domain of the variable

The relevant output is created in a way as to be interpreted without the knowledge of the theoretical methodology. The criterion for injectivity of B that is tested and verified is the only requirement to apply Theorem 1 from [4] and formulate the unique solution.

The core part of the code is given in figure 1.

Conclusion

The computer codes proposed provide 1) formulation of the exact solution of Fredholm integro-differential equations with multipoint or nonlocal integral boundary conditions, 2) solvability exploration of infinite number of examples and 3) immediate construction of operators and functionals within the solution methodology.

```
(*The solution method*)
W := IdentityMatrix[n] - F[g[x]]
V := IdentityMatrix[l] - Ψ[z].nmatrix
inverseA[function_] := 1/(m-1)! ∫_0^t (t-x)^(m-1) * function dx
(*Verify the assumptions of the theorem*)
(*Testing necessary and sufficient conditions for operator Bu=Au-gF(Au) to be injective*)
Det[W]
Det[V]
(*Testing the existence and uniqueness criteria*)
If[Det[W] ≠ 0 && Det[V] ≠ 0, "The IDE has a unique solution", "The solution is not unique"]
(*Here is the unique solution by the exact solution method*)
solution := Simplify[
    inverseA[f[x]] + (inverseA[g[x]] + z.nmatrix.Inverse[V].Ψ[inverseA[g[x]]]).
      Inverse[W].F[f[x]] + z.nmatrix.Inverse[V].Ψ[inverseA[f[x]]]]
Print["The exact solution of the IDE is" Flatten[solution]]
Plot[solution[[1]], {t, 0, 1}, AxesLabel → {"t", "u(t)"}, PlotLabel → "u(t) over the domain [0,1]"]
```

FIGURE 1. The exact solution routine in *Mathematica*

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Combinatorial encoding of continuous processes

Anatoly Vershik

How to encode with locally finite arrays the pure continious message? The simplest example of such problem — the encoding of the sequence of independent random values with contniuous distribuion using discrete locall finite alphabet. This problem is simultaneously related to information theory,combinatorics, dynamics and logic. One of aspects — to enlarge the framework of symbolic dynamics and notion of ordinary stationarity. We give the main example: -how to use Weyl simplices for such encoding and then general definition of quasi-stationarity of the space of paths of the graded graphs.

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Computer algebra aided generation of a mimetic difference scheme for 3D steady Stokes flow

Vladimir P. Gerdt, Yury A. Blinkov and Xiaojing Zhang

In paper [1] two of us suggested an algorithmic approach to generation of finite difference schemes for polynomial nonlinear differential equations on regular grids and applied it in [2] to generation of difference schemes for 2D incompressible Navier-Stokes equations. Then, in [3, 4] the novel concept of s(trong)-consistency which strengthens the universally adopted concept of consistency for difference schemes was introduced and in [5, 6, 7] for 2D incompressible Navier-Stokes equations it was shown that s-consistent schemes have better numerical behavior than the s-inconsistent ones. In addition, in [8] and [9] for steady 2D and 3D Stokes flow, respectively, it was demonstrated that the concept of s-consistency plays the key role in construction of the modified equation.

In the present talk we consider algorithmic issues of computer algebra aided generation of a mimetic difference scheme for the governing system of linear partial differential equations for steady Stokes flow whose involutive form is given by

$$\begin{cases} F^{(1)} := & u_x + v_y + w_z = 0\,, \\ F^{(2)} := & p_x - \frac{1}{\mathrm{Re}}\left(u_{xx} + u_{yy} + u_{zz}\right) - f^{(1)} = 0\,, \\ F^{(3)} := & p_y - \frac{1}{\mathrm{Re}}\left(v_{xx} + v_{yy} + v_{zz}\right) - f^{(2)} = 0\,, \\ F^{(4)} := & p_z - \frac{1}{\mathrm{Re}}\left(w_{xx} + w_{yy} + w_{zz}\right) - f^{(3)} = 0\,, \\ F^{(5)} := & p_{xx} + p_{yy} + p_{zz} - f^{(1)}_x - f^{(2)}_y - f^{(3)}_z = 0\,. \end{cases} \tag{1}$$

where x, y, z are the independent variables; the velocities u, v and w, the pressure p, and the external forces $f^{(1)}$,$f^{(2)}$ and $f^{(3)}$ are the dependent variables; the constant Re is the Reynolds number and $\Delta := \partial_{xx} + \partial_{yy} + \partial_{zz}$ is the Laplace operator. The equation $F^{(5)} = 0$ is integrability condition to Eqs. $\{F^{(i)} = 0 \mid 1 \le i \le 4\}$ called the pressure Poisson equation.

The system (1) possesses the permutational symmetry

$$\{x, u, f^{(1)}\} \longleftrightarrow \{y, v, f^{(2)}\} \longleftrightarrow \{z, w, f^{(3)}\}\,. \tag{2}$$

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To preserve this symmetry at the discrete level we consider the Cartesian grid with the spacing h and apply the method of paper [1] to construct a difference scheme for (1) which is symmetric under permutation (2), strongly consistent and conservative. The s-consistency means inheritance by the scheme such important algebraic property of (1) as vanishing of any element in the differential ideal generated by the polynomials in (1) on any common solution to these equations. This means that for the difference scheme any element of the difference ideal generated by the polynomials in the scheme approximates a polynomial in the differential ideal. Besides, we want to have the scheme to be conservative, i.e. having the conservation law properties inherent in (1).

The procedure of the scheme generation described in paper [9] yields the following results

$$
\begin{cases}
\tilde{F}^{(1)} := & \dfrac{u_{j+2,\,k+1,\,l+1} - u_{j,\,k+1,\,l+1}}{2h} + \dfrac{v_{j+1,\,k+2,\,l+1} - v_{j+1,\,k,\,l+1}}{2h} \\
& + \dfrac{w_{j+1,\,k+1,\,l+2} - v_{j+1,\,k+1,\,l}}{2h} = 0\,, \\
\tilde{F}^{(2)} := & \dfrac{p_{j+2,\,k+1,\,l+1} - p_{j,\,k+1,\,l+1}}{2h} - \dfrac{1}{\mathrm{Re}}\Delta_1\left(u_{j,k,l}\right) - f^{(1)}_{j+1,\,k+1,\,l+1} = 0\,, \\
\tilde{F}^{(3)} := & \dfrac{p_{j+1,\,k+2,\,l+1} - p_{j+1,\,k,\,l+1}}{2h} - \dfrac{1}{\mathrm{Re}}\Delta_1\left(v_{j,k,l}\right) - f^{(2)}_{j+1,\,k+1,\,l+1} = 0\,, \\
\tilde{F}^{(4)} := & \dfrac{p_{j+1,\,k+1,\,l+2} - p_{j+1,\,k+1,\,l}}{2h} - \dfrac{1}{\mathrm{Re}}\Delta_1\left(w_{j,k,l}\right) - f^{(3)}_{j+1,\,k+1,\,l+1} = 0\,, \\
\tilde{F}^{(5)} := & -\dfrac{f^{(1)}_{j+3,\,k+2,\,l+2} - f^{(1)}_{j+1,\,k+2,\,l+2}}{2h} - \dfrac{f^{(2)}_{j+2,\,k+3,\,l+2} - f^{(2)}_{j+2,\,k+1,\,l+2}}{2h} \\
& - \dfrac{f^{(3)}_{j+2,\,k+2,\,l+3} - f^{(3)}_{j+2,\,k+2,\,l+1}}{2h} + \Delta_2\left(p_{j,k,l}\right) = 0\,,
\end{cases}
$$

where where Δ_1 and Δ_2 are finite difference discretizations of the Laplace operator acting on a grid function $g_{j,\,k,\,l}$ as

$$
\begin{aligned}
\Delta_1\left(g_{j,\,k,\,l}\right) := & \quad \frac{g_{j+2,\,k+1,\,l+1} + g_{j+1,\,k+2,\,l+1} + g_{j+1,\,k+1,\,l+2} - 6g_{j+1,\,k+1,\,l+1}}{h^2} \\
& + \frac{g_{j+1,\,k,\,l+1} + g_{j,\,k+1,\,l+1} + g_{j+1,\,k+1,\,l}}{h^2}\,, \\
\Delta_2\left(g_{j,\,k,\,l}\right) := & \quad \frac{g_{j+4,\,k+2,\,l+2} + g_{j+2,\,k+4,\,l+2} + g_{j+2,\,k+2,\,l+4} - 6g_{j+2,\,k+2,\,l+2}}{4h^2} \\
& + \frac{g_{j+2,\,k,\,l+2} + g_{j,\,k+2,\,l+2} + g_{j+2,\,k+2,\,l}}{4h^2}\,.
\end{aligned}
$$

Note, that the replaces Δ_2 with Δ_1 preserves consistency of the scheme but violate its s-consistency. Based on the s-consistent scheme one can compute its modified equation [9] which allows to analyze the order of accuracy of the scheme. The obtained scheme is of the second order.

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Sequential construction of samples from residual allocation models

Yuri Yakubovich

Abstract. We describe a simple procedure to construct a sample from a residual allocation model without sampling the underlying distribution. The talk is based on an ongoing joint work with Jim Pitman, University of California at Berkeley.

Introduction

Samples from random discrete distributions occur in a variety of models of mathematical genetics, ecology, computer science and statistics [1, 3, 6, 7]. Let $(P_1, P_2, \ldots)$ be a random discrete distribution on the positive integers, with $P_j > 0$ for all $j \geq 1$, and $\sum_j P_j = 1$ almost surely. By a sample from (P_j) we understand a sequence of random variables $(X_1, \ldots, X_n)$ such that, given (P_j), they are independent and

$$\mathbb{P}[X_i = j|(P_j)] = P_j, \qquad i = 1, \ldots, n,\ j \in \mathbb{N} = \{1, 2, \ldots\}.$$

A sequence (P_j) can always be represented in the stick-breaking form

$$P_j = H_j \prod_{i=1}^{j-1} (1 - H_i) \tag{1}$$

for some sequence of random discrete hazard probabilities (H_i), with $0 < H_i < 1$ almost surely. Hazards H_j can be interpreted as conditional probabilities $\mathbb{P}[X_i = j|X_i \geq j]$. We say that the random discrete distribution is generated by *residual allocation model (RAM)* if random discrete hazards H_i in (1) are independent. The case when hazards are also identically distributed is also known as *Bernoulli sieve* [2]. The most studied RAM is the celebrated GEM(θ) distribution and its two-parameter generalization [5] which appears for H_i with beta$(1 - \alpha, \theta + i\alpha)$ distribution, that is with density $B(1 - \alpha, \theta + i\alpha)^{-1} h^{-\alpha}(1 - h)^{\theta + i\alpha - 1}$, $0 < h < 1$, for $\alpha \in [0, 1)$ and $\theta > -\alpha$.

There is a natural way to generate samples from a RAM related to the so-called Kingman paintbox construction. First generate a sequence (P_j) by some means, say using the representation (1), and let $F_k := \sum_{j=1}^{k} P_j$ be cumulative sums so that

$$F_0 = 0; \qquad F_k = 1 - \prod_{i=1}^{k}(1 - H_i), \qquad k \in \mathbb{N}, \tag{2}$$

and $P_k = F_k - F_{k-1}$ for $k \geq 1$. Note that

$$0 < F_1 < F_2 < \cdots \uparrow 1 \text{ almost surely.} \tag{3}$$

These points divide the interval $[0, 1]$ on infinite number of subintervals. Next generate n uniform samples $U_1, \ldots, U_i$ on $[0, 1]$ and define $X_i = j$ if $U_i \in [F_{j-1}, F_j)$.

This procedure is effective but requires a construction of interval partition. Our aim here is to describe an alternative approach which does not rely on interval partition at all.

Consecutive construction of a sample

We shall describe the sample $X_1, \ldots, X_n$ in terms of counts

$$N_{n:j} = \#\{i \in \{1, \ldots, n\} : X_i = j\}, \qquad j \in \mathbb{N}. \tag{4}$$

For many characteristics of the sample this is a sufficient statistics, because given distinct sample values $1 \leq x_1 < \cdots < x_k$ and counts $\nu_1, \ldots, \nu_k$ of these values, so that

$$N_{n:x_i} = \nu_i \text{ for } i = 1, \ldots, k, \quad \text{and } N_{n:j} = 0 \text{ for } j \notin \{x_1, \ldots, x_k\},$$

each particular sample with these counts appears with the probability $\binom{n}{\nu_1, \ldots, \nu_k}^{-1}$. A notation for cumulative sums of counts will be also useful:

$$S_{n:j} = \sum_{i \geq j} N_{n:i} = \#\{i \in \{1, \ldots, n\} : X_i \geq j\}. \tag{5}$$

Note that $S_{n:1} = n$.

Introduce binomial moments

$$\mu_j(n, m) = \mathbb{E} H_j^n (1 - H_j)^m. \tag{6}$$

Then the sample can be constructed by the following procedure.

Theorem 1. *Suppose that for we have already generated first $n = 0, 1, \ldots$ values of a sample which have counts $(N_{n:j})$ as in (4). Then in order to learn a value of X_{n+1} one should perform a sequence of choices whether $X_{n+1} = j$, for $j = 1, 2, \ldots$, with the probability to stop at j*

$$\mathbb{P}[X_{n+1} = j | (N_{n:i})_{i \in \mathbb{N}}, X_{n+1} \geq j] = \frac{\mu_j(N_{n:j} + 1, S_{n:j})}{\mu_j(N_{n:j}, S_{n:j})}. \tag{7}$$

For a RAM subject to assumptions (3) this sequence stops at finite step almost surely.

This consecutive construction works best when there are explicit formulas for the moments (6). This is the case when H_i have beta distribution, as in the GEM model [4].

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Factorial Polynomials in Computer Algebra Problems Related to Symbolic Summation

Eugene V. Zima

Abstract. We consider a natural succinct representation for factorial polynomials along with the set of low complexity lazy manipulation and evaluation rules. This leads to immediate improvements of the worst case running-time complexity of many basic steps in standard algorithms for indefinite and definite summation. Applications of this technique to computation of rational normal forms and anti-differences of hypergeometric terms is discussed along with a prototype Maple implementation.

Introduction

One of important intermediate representations arising in algorithms of symbolic summation is the Gosper-Petkovšek form of a rational function [1, 3] The modern term for this representation is *Polynomial Normal Form* (PNF for short [1]). One of the problems with PNF is that the degree of one of the polynomials forming the PNF can be exponential in the size of the numerator and the denominator of the input rational function. This together with the fact that the degree of PNF drives the running time complexity of summation algorithms directly influences efficiency of standard summation algorithms in computer algebra systems.

This contribution is based on very simple observation that PNF can be represented and manipulated succinctly and lazily, reducing the running time complexity of standard tasks involved in algorithms of symbolic summation. This succinct representation is natural, in a sense that it is explicit, as opposed to the idea of implicit representation described in [2].

1. Preliminaries

Let $\mathbb{K}$ be a field of characteristic zero, x – an independent variable, E – the shift operator with respect to x, i.e., $Ef(x) = f(x+1)$ for an arbitrary $f(x)$. Consider $R \in \mathbb{K}(x)$. If $z \in \mathbb{K}$ and monic polynomials $A, B, C \in \mathbb{K}[x]$ satisfy
(i) $R = z \cdot \frac{A}{B} \cdot \frac{EC}{C}$,
(ii) $A \perp E^k B$ for all $k \in \mathbb{N}$,
then (z, A, B, C) is a *polynomial normal form* (PNF) of R. If in addition,

(iii) $A \perp C$ and $B \perp EC$,
then (z, A, B, C) is a *strict* (PNF) of R (see [1] for details).

For example, for the rational function $\frac{x+1000}{x+1}$ PNF is

$$1, 1, 1, (x+999) \cdot (x+998) \cdot (x+997) \cdot \cdots \cdot (x+2) \cdot (x+1),$$

with polynomial C of degree 999.

An important notion widely used in the context of algorithmic summation is the *dispersion set* of polynomials $p(x)$ and $q(x)$, which is the set of positive integers h such that $\deg(\gcd(p(x+h), q(x))) > 0$. Another important notion is the largest element of the dispersion set known as the *dispersion*. One more piece of standard terminology required here is the notion of *shift equivalence* of polynomials: two polynomials $u(x), v(x) \in \mathbb{K}[x]$ are shift equivalent if there exists $h \in \mathbb{Z}$, such that $u(x+h) = v(x)$. Finally, following [4] define the *factorial polynomial* (a generalization of the falling factorial) for $p(x) \in \mathbb{K}[x]$ as

$$[p(x)]_k = p(x) \cdot p(x-1) \cdot \ldots \cdot p(x-k+1) \tag{1}$$

for $k > 0$ and $[p(x)]_0 = 1$. Observe, that factorial polynomials naturally appear in the last component of PNF, assuming that the dispersion of the numerator and denominator of a given rational function is nonzero.

2. Succinct representation of factorial polynomials and lazy manipulation rules

We first note, that the left hand side of (1) offers succinct (most compact) representation of the product in the right hand side for large values of k, as it requires $\Theta(\log k)$ bits to represent the polynomial $p(x) \cdot p(x-1) \cdot \ldots \cdot p(x-k+1)$ assuming the degree of $p(x)$ is fixed. The same polynomial would require $\Theta(k \log k)$ bits if represented as in [2]. For example, for $R = \frac{(x+10)^2 (2x+29)}{(x+1)(2x-1)(5x+1)}$, the PNF in succinct form is

$$\left(1/5, x+10, x+1/5, [x+9]_9 \left[x + \frac{27}{2}\right]_{15}\right),$$

which is much shorter compared to the expanded representation of the degree 24 polynomial C. Factorial polynomials satisfy many obvious identities, which capture their multiplicative nature and allow manipulate them without expanding. We list only few of them for illustration purposes:

$$[p(x)]_k = [p(x)]_{k-1} \cdot p(x-k+1), \quad [p(x+1)]_k = p(x+1) \cdot [p(x)]_{k-1}, \quad \text{for } k > 0,$$
$$[p_1(x) \cdot p_2(x)]_k = [p_1(x)]_k \, [p_2(x)]_k \quad \textit{etc.}$$

Based on these it is easy to implement lazy evaluation rules, such as for example,

$$A \cdot [p(x)]_k \pm B \cdot [p(x+1)]_k = [A \cdot p(x-k+1) \pm B \cdot p(x+1)] \cdot [p(x)]_{k-1}, \tag{2}$$

which holds for arbitrary expressions A and B. Another set of rules involves computation of gcd and cancelations. For example, given natural h, k, and l:

$$\gcd([p(x)]_k, [p(x+h)]_l) = \begin{cases} 1 & \text{if } l-h \le 0, \\ [p(x)]_{\min(k,l-h)} & \text{otherwise.} \end{cases}$$

The ultimate goal of lazy manipulation rules is to avoid complete expansion of the involved factorial polynomials as much as possible.

3. Applications

The natural succinct representation together with above mentioned rules offer immediate improvement to the worst case running-time complexity of many basic steps in standard algorithms for indefinite and definite summation.

3.1. Rational Normal Forms

The notion of Rational Normal Form (RNF) was introduced in the context of hypergeometric summation (see [1] for the definition). The main steps in computation of RNF for a given rational function $R \in \mathbb{K}(x)$ involve construction of two PNFs, computation of gcd, and divisions:

```
(z,a,b,c) := PolynomialNormalForm(R,n);
(z1,a1,b1,c1) := PolynomialNormalForm(b/a,n);
g := gcd(c,c1);
return (z,b1,a1,c/g,c1/g)
```

The gain from using succinct representation is transparent from the following example: for $R = \frac{x(x+1000)}{(x+3)(x+1003)}$ the first call to PolynomialNormalForm produces $(1, x, x+1003, [x+999]_{997})$ with polynomial c of degree 997, the second call produces $(1, 1, 1, [x+1002]_{1003})$ with polynomial $c1$ of degree 1003, and $\gcd(c, c1) = [x+999]_{997}$. The RNF for R is

$$(1, 1, 1, 1, (x+1001)\,(x+1002)\,(x+1)\,(x+2)\,x\,(x+1000)),$$

with most of the terms from PNFs cancelled. Our prototype produces this result in 0.012 seconds, while standard Maple implementation requires 3.5 second on the same computer.

3.2. Gosper algorithm for summable hypergeometric terms

Recall that a nonzero expression $F(x)$ is called a hypergeometric term over $\mathbb{K}$ if there exists a rational function $r(x) \in \mathbb{K}(x)$ such that $F(x+1)/F(x) = r(x)$. Usually $r(x)$ is called the rational *certificate* of $F(x)$. The problem of indefinite hypergeometric summation (anti-differencing) is: given a hypergeometric term $F(x)$ to find a hypergeometric term $G(x)$, which satisfies the first order linear difference equation

$$(E-1)G(x) = F(x). \tag{3}$$

If found, write $\sum_x F(x) = G(x) + c$, where c is an arbitrary constant.

In [3] Gosper described a decision procedure, which starts with converting the rational certificate of the given summand to the Gosper-Petkovšek form (z, A, B, C) reducing the search for the sum to the search of a polynomial $y(x)$ solving the *key equation*:

$$A(x)y(x+1) - B(x-1)y(x) = C(x). \tag{4}$$

If $y(x)$ is found, then

$$G(x) = F(x)\frac{B(x-1)y(x)}{C(x)}. \tag{5}$$

Sometimes the polynomial $C(x)$ from (4) is called *the universal denominator*. One well-known problem with Gosper's algorithm is that the universal denominator can have pessimistically large degree (i.e., $C(x)$ and $y(x)$ can have very large degree common factor, which will cancel after substituting the solution $y(x)$ into (5)). This in turn can lead to the exponential dependency of the running time on the size of the input, even when input and output is small. Using the succinct representation of PNF along with multiplicative properties of the solution of (5) and of the factorial polynomials it is possible to remove extraneous terms in this universal denominator before solving the key equation. This leads to improvements in the running time complexity of Gosper's algorithm in case of summable non-rational hypergeometric summands with large dispersion of the rational certificate.

Implementations of all the above mentioned techniques in Maple are compared to standard Maple summation tools and show not only theoretical but also practical improvements.

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