On the Moments of Squared Binomial Coefficients

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Abstract. Explicit recurrent formulas for ordinary and alternated power moments of the squared binomial coefficients are derived in this article. Every such moment proves to be a linear combination of the previous ones via a coefficient list of the relevant Krawtchouk polynomial.

Introduction

In this article, we study the sums of form

$$\mu_r^{(n)} = \sum_{m=0}^{n} m^r \binom{n}{m}^2 \text{ and } \nu_r^{(n)} = \sum_{m=0}^{n} (-1)^m m^r \binom{n}{m}^2, n \geq 0 \tag{1}$$

and refer them as the r-th order (where $r \geq 0$ and $0^0 = 1$ by convention) ordinary and alternating moments of the squared binomial coefficients.

Such sums emerge very often in different theoretical and applied mathematical areas, for example, see A000984, A002457, A037966, A126869, A100071, and A294486 in Sloane's database OEIS of integer sequences.

Unfortunately, up to the date (February 2020) not many explicit and closed formulas for these sums are known and moreover all such formulas are limited in order by $r \leq 4$, see [2], [3], [6], and A074334.

In the present article we prove two main theorems and their corollaries (in the Sections 3 and 4), providing the explicit recurrent formulas for obtaining the closed forms for the aforesaid moments of any order $r \geq 0$. We also give some examples of applications of these results.

1. Introduction

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2. Preliminaries

2.1. Hadamard transform, Krawtchouk polynomials and McWilliams duality formula

In this section, we remind to the reader some definitions and results from algebraic coding theory. The proofs of the claims can be found for example in [4].

For any integer $n \geq 1$, let $\mathbb{F}_2^n$ be an n-dimensional vector space over the binary field $\mathbb{F}_2 = \{0, 1\}$ and let V_n be the 2^n-dimensional Euclidean vector space of all functions $f\colon \mathbb{F}_2^n \to \mathbb{R}$ equipped with the usual scalar product $(f, g) = \sum_{\boldsymbol{v}\in\mathbb{F}_2^n} f(\boldsymbol{v})g(\boldsymbol{v})$

Every additive character of the space $\mathbb{F}_2^n$ can be written in a form $\chi_{\boldsymbol{u}}(\boldsymbol{v}) = \chi_{\boldsymbol{v}}(\boldsymbol{u}) = (-1)^{\boldsymbol{v}\cdot\boldsymbol{u}}$, where the dot product is defined as $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^n u_i v_i = |\boldsymbol{u} \cap \boldsymbol{v}|$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are interpreted (in the obvious way) as subsets of indices $\{1, 2, ..., n\}$. Note that for the vector $\mathbf{1} = (1, 1, \ldots, 1) \in V_n$ the scalar product $(\boldsymbol{v}, \mathbf{1}) = |\boldsymbol{v} \cap \{1, 2, ..., n\}|$ is $|\boldsymbol{v}|$, the Hamming weight of $\boldsymbol{v}$.

It is a well-known fact that the full set of characters forms an orthogonal basis in the space V_n for which

$$(\chi_{\boldsymbol{u}}, \chi_{\boldsymbol{v}}) = 2^n \delta_{\boldsymbol{u},\boldsymbol{v}} = 2^n \begin{cases} 1 & \text{if } \boldsymbol{u} = \boldsymbol{v} \\ 0 & \text{if } \boldsymbol{u} \neq \boldsymbol{v} \end{cases} \tag{3}$$

Definition 1. *For every function $f \in V_n$ we define the Hadamard transform as*

$$\widehat{f}(\boldsymbol{u}) = (f, \chi_{\boldsymbol{u}}) = \sum_{\boldsymbol{v}\in V_n} (-1)^{|\boldsymbol{v}\cap\boldsymbol{u}|} f(\boldsymbol{v}). \tag{4}$$

Note that $(\widehat{f}, \widehat{g}) = 2^n (f, g)$ and $\widehat{\widehat{f}} = 2^n f$ (unitary and involutory properties of Hadamard transform).

Define now for every $0 \le r \le n$ the r-th weight-function $\psi_r(\boldsymbol{v}) = \begin{cases} 1 & \text{if } |\boldsymbol{v}| = r \\ 0 & \text{if } |\boldsymbol{v}| \ne r \end{cases}$ and let $\mathfrak{H}_r^{(n)}$ be the r-th Hamming sphere in $\mathbb{F}_2^n$: $\mathfrak{H}_r^{(n)} = \{\boldsymbol{v} : \psi_r(\boldsymbol{v}) = 1\}$.

Definition 2. *For any $f \in V_n$ the $(n+1)$-tuple of real numbers $\mathfrak{S}(f) = (S_0(f), S_1(f), \ldots S_n(f))$, where $S_r(f) = (f, \psi_r) = \sum_{\boldsymbol{v} \in \mathfrak{H}_r^{(n)}} f(\boldsymbol{v})$ is said to be the weight spectrum of f while $\mathfrak{S}(\widehat{f})$ is referred as a dual weight spectrum of f .*

Due to the unitary and involutory properties of the Hadamard transform one has:

$$S_r(\widehat{f}) = (\widehat{f}, \psi_r) = 2^{-n}(\widehat{\widehat{f}}, \widehat{\widehat{\psi}}_r) = (f, \widehat{\psi}_r). \tag{5}$$

Functions $\widehat{\psi}_r(\boldsymbol{v})$ depend obviously only on $|\boldsymbol{v}|$ and this gives rise to the following definition:

Definition 3. *Function*

$$\widehat{\psi}_r(\boldsymbol{v}) = \widehat{\psi}_r(|\boldsymbol{v}|) = K_r^{(n)}(x) = \sum_{i=0}^{r} (-1)^i \binom{n-x}{r-i} \binom{x}{i} \tag{6}$$

where $x = |\boldsymbol{v}|$ is being called as the r-th Krawtchouk polynomial of order n; $0 \le r = \deg K_r^{(n)} \le n$.

Now we can easily rewrite (5) to obtain the famous MacWilliams formula for dual spectrae: $S_r(\widehat{f}) = \sum_{i=0}^{n} K_r^{(n)}(i) S_i(f)$.

2.2. Auxiliary lemmata

We will use these results in the sequel, but they may have independent combinatorial interest, as well.

Lemma 1. *For any nonnegative integer d, $d \ge 1$ let $g(\boldsymbol{v}) = \binom{d}{|\boldsymbol{v}|}$, where $\boldsymbol{v} \in V_n$. Then $\widehat{g}(\boldsymbol{u}) = K_d^{(n+d)}(|\boldsymbol{u}|), \quad \boldsymbol{u} \in V_n$*

Lemma 2. *For every integer $k, 0 \le k \le d$ the following identity is valid:*

$$\sum_{i=0}^{n} K_i^{(n)}(k) K_d^{(n+d)}(i) = 2^n \binom{d}{k}. \tag{7}$$

Corollary 1.

$$(-2)^n \sum_{i=0}^{n} \binom{\frac{i-1}{2}}{n} K_i^{(n)}(k) = \binom{n}{k}. \tag{8}$$

Lemma 3.

$$\sum_{m=0}^{n} \binom{n}{m}^2 K_j^{(n)}(m) = \binom{n}{j} K_n^{(2n)}(j) \tag{9}$$

for every nonnegative integer j.

Remark 1. *For $j = 0$ we get from* (9) *the classical identity*

$$\mu_0^{(n)} = \sum_{m=0}^{n} \binom{n}{m}^2 = \binom{2n}{n}. \tag{10}$$

3. Recurrent formula for the moments $\mu_j^{(n)}$ and its applications

For a fixed nonnegative integer j let $\left(\kappa_r^{(n,j)}\right)_{0\le r\le j}$ be a coefficient list of the polynomial $K_j^{(n)}$ and let $\boldsymbol{\mu} = (\mu^s)_{s\ge 0}$ be an *umbral* variable with a rule $\mu^s \rightleftarrows \mu_s^{(n)}, s \ge 0$.

Remark 2. *Descriptions of umbral calculus can be found in many sources. We recommend the reader to consult* [8]*, for instance. A more developed and formalized treatises are available at* [5] *and* [7]*.*

In this article, we need only the elementary notion of umbral variable as a linear functional $\to$ on $\mathbb{C}[[\mu]]$ (the formal power series over μ) defined as $\to(\mu^s) = \mu_s^{(n)}$, $s \ge 0$.

Theorem 1.

$$K_j^{(n)}(\boldsymbol{\mu}) = \binom{n}{j} K_n^{(2n)}(j). \tag{11}$$

Corollary 2. *(Recurrent formula for $\mu_j^{(n)}$)*

$$\frac{(-2)^j}{j!}\mu_j^{(n)} = \binom{n}{j} K_n^{(2n)}(j) - \sum_{r=0}^{j-1} \kappa_r^{(n,j)} \mu_r^{(n)}, \quad j \ge 1, \mu_0^{(n)} = \binom{2n}{n}. \tag{12}$$

For example, for $j = 6$ one can consecutively applying Corollary 2 find

$$\mu_6^{(n)} = \frac{n^3\left(n^6 + 3n^5 - 13n^4 - 15n^3 + 30n^2 + 8n - 2\right)}{8(2n-1)(2n-3)(2n-5)} \binom{2n}{n}.$$

4. Recurrent formula for the alternating moments $\nu_j^{(n)}$ and their applications

The case of the alternated moments is in general similar to the previous one but is more subtle: Let $\boldsymbol{\nu} = (\nu^s)_{s\ge 0}$ be an umbral variable with a rule $\nu^s \rightleftarrows \nu_s^{(n)}, s \ge 0$.

Theorem 2.

$$K_j^{(n)}(\boldsymbol{\nu}) = \binom{n}{j} K_n^{(2n)}(n-j). \tag{13}$$

Corollary 3. *(Recurrent formula for $\nu_j^{(n)}$)*

$$\frac{(-2)^j}{j!}\nu_j^{(n)} = \binom{n}{j} K_n^{(2n)}(n-j) - \sum_{r=0}^{j-1} \kappa_r^{(n,j)} \nu_r^{(n)}, \quad j \geq 1, \tag{14}$$

$$\nu_0^{(n)} = \begin{cases} (-1)^{\frac{n}{2}} \binom{n}{n/2} & \textit{if } n \equiv 0 \textit{ (mod 2)} \\ 0 & \textit{if } n \equiv 1 \textit{ (mod 2)} \end{cases}. \tag{15}$$

For example, for $j = 6$, one can consecutively applying corollary 3 find:

$$\nu_6^{(n)} = \begin{cases} (-1)^{\frac{n+2}{2}} \frac{n^3(n+1)(3n-1)}{8} \binom{n}{n/2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{(-1)^{\frac{n-1}{2}} n^2(n+1)(n^3+n^2-9n+3)}{8} \binom{n}{(n+1)/2} & \text{if } n \equiv 1 \pmod 2 \end{cases}. \tag{16}$$

References

[1] Gogin N., Hirvensalo M.: *Recurrent Construction of MacWilliams and Chebyshev Matrices.* Fundamenta Informaticae 116:1–4, pp. 93–110 (2012). DOI: 10.3233/FI-2012-671

[2] Gould H. W.: *Combinatorial Identities: A Standardized Set of Tables Listing 500 Binomial Coefficient Summations.* Morgantown Printing and Binding Co (1972).

[3] Graham R.L., Knuth D. E., Patashnik O.: *Concrete Mathematics.* Addison-Wesley (1994).

[4] MacWilliams F.J., Sloane N. J. A.: *The Theory of Error-Correcting Codes.* North Holland (1977).

[5] Licciardi, S.: *Umbral Calculus, A Different Mathematical Language.* `https://arxiv.org/abs/1803.03108` (2018) (retrieved February 15, 2020).

[6] Riordan J.: *Combinatorial Identities.* John Wiley & Sons Inc. (1968).

[7] Nigel Ray: *Universal Constructions in Umbral Calculus.* In: Bruce E. Sagan, Richard P. Stanley (Eds.): Mathematical Essays in honor of Gian-Carlo Rota, pp. 343–357. Springer (1996).

[8] Roman, S.M., Rota G.-C.: *The Umbral Calculus.* Advances in Mathematics 27, pp. 95–188 (1978).

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