The close relation between border and Pommaret marked bases

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Abstract. Given a finite order ideal $\mathcal{O}$, we investigate border and Pommaret marked sets related to this order ideal. We use the framework of reduction structures given in [3].
First, we prove that a marked set B on the border of $\mathcal{O}$ is a basis if and only if the marked set on the Pommaret basis of the complementary ideal of $\mathcal{O}$ contained in B is a basis and generates the same ideal as B.
As a byproduct, using a functorial description of border and Pommaret marked bases, we obtain that the scheme parameterizing marked bases on the border of $\mathcal{O}$ and the scheme parameterizing marked bases on the Pommaret basis of the complementary ideal of $\mathcal{O}$ are isomorphic. We also explicitly construct such an isomorphism.

Introduction

Consider the variables $x_1, \dots, x_n$, with $x_1 < \cdots < x_n$, the set $\mathbb{T}$ containing the terms in the variables $x_1, \dots, x_n$ and the polynomial ring $R_A := A[x_1, \dots, x_n]$, being A a Noetherian algebra over a field K with unit 1_K.

If $\mathcal{O} \subset \mathbb{T}$ is a finite order ideal, we can define a set $F \subset R_A$ of monic marked polynomials whose *head terms* are the *border of* $\mathcal{O}$, $\partial\mathcal{O}$, and study the conditions ensuring that F is a $\partial\mathcal{O}$-marked basis, i.e. $(F) \oplus \langle \mathcal{O} \rangle_A = R_A$, where $\langle \mathcal{O} \rangle_A$ is the A-module generated by $\mathcal{O}$.

Border marked bases (border bases in the literature) were first introduced in [9] and investigated from a numerical point of view because of their stability with respect to perturbation of the coefficients [10, 11]. Border bases have also attracted interest from an algebraic point of view [7, Section 6.4], also because given a finite order ideal $\mathcal{O}$, the border bases on $\mathcal{O}$ parameterize an open subset of a Hilbert scheme (see also [5, 6]).

Since every Artinian monomial ideal in R_K has a *Pommaret basis*, given a finite order ideal $\mathcal{O}$, we can consider the Pommaret basis $\mathcal{P}_\mathcal{O}$ of the monomial ideal generated by $\mathbb{T} \setminus \mathcal{O}$ and construct monic marked sets and bases whose head terms form $\mathcal{P}_{\mathcal{O}}$.

Marked bases on strongly stable ideals were first introduced in [4] with the aim to parameterize open subsets of a Hilbert scheme, in order to study it locally. This kind of basis does not need any finiteness assumption on the underlying order ideal. In [1] marked bases were considered only in the case of homogeneous polynomials, but in [2] also non-homogeneous marked bases over monomial ideals having a Pommaret basis were considered, in order to have more efficient computational techniques for the homogeneous case.

The goal of our work is comparing marked sets (and bases) on the border $\partial\mathcal{O}$ of $\mathcal{O}$ and on the Pommaret basis $\mathcal{P}_{\mathcal{O}}$ of the ideal $(\mathbb{T} \setminus \mathcal{O})$. To this aim we use the framework of reduction structures [3], and a functorial approach to study the schemes parameterizing these two different bases. The monicity of the marked sets we consider is crucial for the use of functors.

Observing that a set B of marked polynomials on $\partial\mathcal{O}$ always contains a set P of marked polynomials on $\mathcal{P}_{\mathcal{O}}$, we prove that B is a $\partial\mathcal{O}$-marked basis if and only if P is a $\mathcal{P}_{\mathcal{O}}$-marked basis and generates the same ideal as B.

As a byproduct, using a functorial description of border and Pommaret marked bases, we obtain that the scheme parameterizing $\partial\mathcal{O}$-marked bases and the scheme parameterizing $\mathcal{P}_{\mathcal{O}}$-marked bases are isomorphic. We also explicitly construct such an isomorphism.

1. Framework

If σ is a term in $\mathbb{T}$, we denote by $\min(\sigma)$ the smallest variable dividing σ. A set $\mathcal{O}$ of terms in $\mathbb{T}$ is called an *order ideal* if for every $\sigma \in \mathbb{T}$ and every $\tau \in \mathcal{O}$, if σ divides τ, then σ belongs to $\mathcal{O}$.
Given a finite order ideal $\mathcal{O}$, *the border* of $\mathcal{O}$ is $\partial\mathcal{O} := \{x_i \cdot \tau \mid \tau \in \mathcal{O}, i \in \{1, \dots, n\}\} \setminus \mathcal{O}$ [7, Definition 6.4.4], and the *Pommaret basis* of $\mathbb{T} \setminus \mathcal{O}$ is $\mathcal{P}_{\mathcal{O}} = \{\sigma \in \mathbb{T} \setminus \mathcal{O} | \sigma/\min(\sigma) \in \mathcal{O}\}$. Observe that $\mathcal{P}_{\mathcal{O}} \subset \partial\mathcal{O}$.

Definition 1. [3, Definition 3.1] A *reduction structure* $\mathcal{J}$ in $\mathbb{T}$ is a 3-uple $\mathcal{J} := (\mathcal{H}, \mathcal{L} := \{\mathcal{L}_\alpha \mid \alpha \in \mathcal{H}\}, \mathcal{T} := \{\mathcal{T}_\alpha \mid \alpha \in \mathcal{H}\})$ where: $\mathcal{H} \subseteq \mathbb{T}$ is a *finite set* of terms; for every $\alpha \in \mathcal{H}$, $\mathcal{T}_\alpha \subseteq \mathbb{T}$ is an order ideal, such that $\cup_{\alpha \in \mathcal{H}} \{\tau\alpha | \tau \in \mathcal{T}_\alpha\} = (\mathcal{H})$; for every $\alpha \in \mathcal{H}$, $\mathcal{L}_\alpha$ is a finite subset of $\mathbb{T} \setminus \{\tau\alpha | \tau \in \mathcal{T}_\alpha\}$.

A *marked polynomial* is a polynomial $f \in R_A$ with a specified term of $\mathrm{Supp}(f)$, the *head term* of f, denoted by $\mathrm{Ht}(f)$, which appears in f with coefficient 1_K.

Definition 2. [3, Definitions 4.2 and 4.3] Given a reduction structure $\mathcal{J} = (\mathcal{H}, \mathcal{L}, \mathcal{T})$, a set F of exactly $|\mathcal{H}|$ marked polynomials in R_A is called a *$\mathcal{H}$-marked set* if, for every $\alpha \in \mathcal{H}$, there is $f_\alpha \in F$ with $\mathrm{Ht}(f_\alpha) = \alpha$ and $\mathrm{Supp}(f) \subseteq \mathcal{L}_\alpha$.
Let $\mathcal{O}_{\mathcal{H}}$ be the order ideal given by the terms of $\mathbb{T}$ outside the ideal generated by $\mathcal{H}$. A $\mathcal{H}$-marked set F is called a *$\mathcal{H}$-marked basis* if $(F) \oplus \langle \mathcal{O}_{\mathcal{H}} \rangle_A = R_A$.

From now on, let the terms of the border $\partial\mathcal{O}$ be ordered by increasing degree (terms of the same degree are ordered arbitrarily) and labelled coherently: for every $\beta_i, \beta_j \in \partial\mathcal{O}$, if $i < j$ then $\beta_i < \beta_j$.

Definition 3. Let $\mathcal{O} \subset \mathbb{T}$ be a finite order ideal.
The *Pommaret reduction structure* $\mathcal{J}_\mathcal{P}$ is the reduction structure with $\mathcal{H} = \mathcal{P}_\mathcal{O}$ and, for every $\alpha \in \mathcal{P}_\mathcal{O}$, $\mathcal{L}_\alpha = \mathcal{O}$ and $\mathcal{T}_\alpha = \mathbb{T} \cap K[x_1, \dots, \min(\alpha)]$.
The *border reduction structure* $\mathcal{J}_{\partial\mathcal{O}}$ is the reduction structure with $\mathcal{H} = \partial\mathcal{O}$ and, for every $\beta_i \in \partial\mathcal{O}$, $\mathcal{L}_{\beta_i} = \mathcal{O}$ and $\mathcal{T}_{\beta_i} = \{\mu \in \mathbb{T} \mid \forall j > i,\ \beta_j \text{ does not divide } \beta_i\mu\}$.

For every reduction structure $\mathcal{J} = (\mathcal{H}, \mathcal{L}, \mathcal{T})$ and every $\mathcal{H}$-marked set F, it is possible to define a *reduction relation* on polynomials in R_A, that we denote by $\to_{F\,\mathcal{J}}$. If B (resp. P) is a $\partial\mathcal{O}$-marked set (resp. a $\mathcal{P}_\mathcal{O}$-marked set), for every $f \in R_A$ there is $h_B \in \langle\mathcal{O}\rangle_A$ (resp. $h_P \in \langle\mathcal{O}\rangle_A$) such that $f \to_{B\,\mathcal{J}_\mathcal{O}} h_B$ (resp. $f \to_{P\,\mathcal{J}_{\mathcal{P}_\mathcal{O}}} h_P$). Observe that in general $h_B \neq h_P$. In particular, both the border reduction and the Pommaret reduction structures give Noetherian and confluent reduction relations. These properties ensure that $(B) + \langle\mathcal{O}\rangle_A = (P) + \langle\mathcal{O}\rangle_A = R_A$.

2. Main results

Theorem 4. *Let $\mathcal{O} \subset \mathbb{T}$ be a finite order ideal. Let B be a $\partial\mathcal{O}$-marked set in R_A and we denote by P the $\mathcal{P}_\mathcal{O}$-marked set contained in B. Then we have*

$$B \text{ is a } \partial\mathcal{O}\text{-marked basis} \Leftrightarrow P \text{ is a } \mathcal{P}_\mathcal{O}\text{-marked basis and } (B) = (P).$$

Definition 5. [3, Appendix A] Let $\mathcal{O} \subset \mathbb{T}$ be a finite order ideal and let $\mathcal{J} = (\mathcal{H}, \mathcal{L}, \mathcal{T})$ be a reduction structure with $(\mathcal{H}) = \mathbb{T} \setminus \mathcal{O}$. We consider the functor

$$\mathcal{M}b_\mathcal{J} : \text{ Noeth-}k\text{-Alg} \longrightarrow \text{ Sets},$$

that associates to every Noetherian k-Algebra A the set $\mathcal{M}b_\mathcal{J}(A)$ consisting of all the ideals $I \subset R_A$ generated by a $\mathcal{H}$-marked basis, and to every morphism of Noetherian k-algebras $\phi : A \to A'$ the morphism $\mathcal{M}b_\mathcal{J}(\phi) : \mathcal{M}b_\mathcal{J}(A) \to \mathcal{M}b_\mathcal{J}(A')$ that operates in the following natural way:

$$\mathcal{M}b_\mathcal{J}(\phi)(I) = I \otimes_A A'.$$

Remark 6. The monicity of marked sets and bases guarantees that marked set and bases are preserved by extension of scalars (see also [3, Lemmas A.1 and A.2]).

If $|\mathcal{O}| = \ell$ and $|\partial\mathcal{O}| = m$, we define $C := \{C_{i,j}\}_{1\le i\le m, 1\le j\le \ell}$. The *generic $\partial\mathcal{O}$-marked set* [8, Definition 3.1] is the set $\mathscr{B}$ of marked polynomials $\{g_1, \dots, g_m\} \subset R_{K[C]}$ with $g_i = \tau_i - \sum_{j=1}^{\ell} C_{ij}\sigma_j$.

The set $\mathscr{B}$ contains the generic $\mathcal{P}_\mathcal{O}$-marked set $\mathscr{P}$. We denote by $\tilde{C}$ the set of parameters not appearing in $\mathscr{P}$. By Buchberger criteria for $\partial\mathcal{O}$-marked bases [7, Proposition 6.4.34] and for $\mathcal{P}_\mathcal{O}$-marked bases [2, Proposition 5.6], it is possible to prove that the functor $\mathcal{M}b_{\mathcal{J}_{\partial\mathcal{O}}}$ (resp. $\mathcal{M}b_{\mathcal{J}_{\mathcal{P}_\mathcal{O}}}$) is the functor of points of $\mathrm{Spec}(K[C]/\mathfrak{B})$ (resp. $\mathrm{Spec}(K[C \setminus \tilde{C}]/\mathfrak{P})$, where $\mathfrak{B}$ (resp. $\mathfrak{P}$) is generated by a finite set of polynomials in $K[C]$ (resp. $K[C\setminus \tilde{C}]$) explicitly computed by $\to_{\mathscr{B},\mathcal{J}_{\partial\mathcal{O}}}$ (resp. $\to_{\mathscr{P}\,\mathcal{J}_{\mathcal{P}_\mathcal{O}}}$). Thanks to Theorem 4, we can prove the following.

Theorem 7.

1. *The schemes* $\mathrm{Spec}(K[C]/\mathfrak{B})$ *and* $\mathrm{Spec}(K[C\setminus \tilde{C}]/\mathfrak{P})$ *are isomoprhic;*
2. *there is a isomorphism* $\psi : \mathrm{Spec}(K[C]/\mathfrak{B}) \to \mathrm{Spec}(K[C\setminus \tilde{C}]/\mathfrak{P})$ *defined by computing for every* $\beta \in \partial\mathcal{O}\setminus\mathcal{P}_\mathcal{O}$ *the polynomial* $h_\beta \in \langle \mathcal{O}\rangle_A$ *such that* $\beta \to_{\mathscr{P}\,\mathcal{J}_{\mathcal{P}_\mathcal{O}}} h_\beta$.

References

[1] C. Bertone, F. Cioffi, P. Lella, and M. Roggero, *Upgraded methods for the effective computation of marked schemes on a strongly stable ideal*, J. Symbolic Comput., 50, (2013).

[2] C. Bertone, F. Cioffi, and M. Roggero, *Macaulay-like marked bases*, J. Algebra Appl., 16, (2017), no.5.

[3] M. Ceria, T. Mora, and M. Roggero, *A general framework for Noetherian well ordered polynomial reductions*, J. Symbolic Comput., 95, (2019).

[4] F. Cioffi, and M. Roggero, *Flat families by strongly stable ideals and a generalization of Gröbner bases*, J. Symbolic Comput., 46, (2011), no.9.

[5] M. E. Huibregtse, *Some syzygies of the generators of the ideal of a border basis scheme*, Collect. Math., 62, (2011), no.3.

[6] M. Lederer, *Gröbner strata in the Hilbert scheme of points*, J. Commut. Algebra, 3, (2011), no.3.

[7] M. Kreuzer, and L. Robbiano, *Computational commutative algebra. 2*, Springer-Verlag, Berlin, (2005).

[8] M. Kreuzer, and L. Robbiano, *Deformations of border bases*, Collect. Math., 59, (2008), no.3.

[9] M.G. Marinari, H.M. Möller, and T. Mora, *Gröbner bases of ideals defined by functionals with an application to ideals of projective points*, Appl. Algebra Engrg. Comm. Comput. 4, (1993), no.2.

[10] B. Mourrain, *A new criterion for normal form algorithms*, Lecture Notes in Comput. Sci.,vol.1719,Springer, Berlin, (1999).

[11] H. J. Stetter, *Numerical polynomial algebra*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (2004).

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