An Effectively Computable Projective Invariant

Alexandr V. Seliverstov
Institute for Information Transmission Problems
of the Russian Academy of Sciences
(Kharkevich Institute)

POLYNOMIAL COMPUTER ALGEBRA
2020

Let us consider generalized register machines over a field of characteristic zero

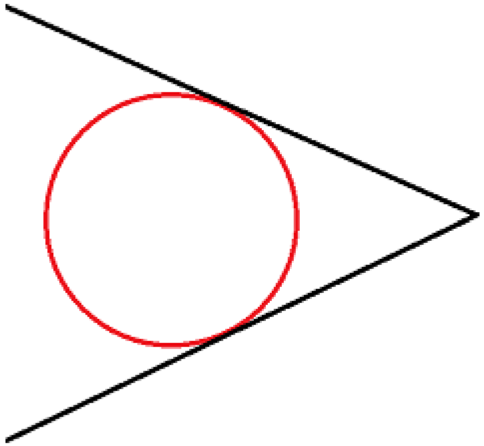
$$(\mathbb{K}, 0, 1, +, -, \times)$$

They are closely related to the machines over reals defined by L. Blum, M. Shub, and S. Smale (1989).
Each register contains an element of $\mathbb{K}$.
There exist index registers containing nonnegative integers.
The running time is said polynomial, when the total number of operations performed before the machine halts is bounded by a polynomial in the number of registers occupied by the input.
Initially, this number is placed in the zeroth index register.

If a polynomial serves as an input, then its coefficients are written into registers. For a sparse polynomial, many registers contain zeros.

A hypersurface in $\mathbb{P}^n$ is the vanishing locus of a form, i.e., a homogeneous polynomial in $n+1$ variables.
It is hard to recognize whether a given cubic hypersurface is smooth.

For a smooth plane curve of order d, in accordance with the Plücker formulae, there exist exactly $d^2 - d$ tangent straight lines passing through the general point of the plane.
Contrariwise, if the exist $d^2 - d$ different tangent straight lines passing through a point, then the plane curve is smooth.



In case $d = 2$, there are two tangent straight lines to a conic section. But in general case, there is no tangent straight line to any singular quadric, which is a pair of straight lines. There is only double straight line passing through the double point.

Let us consider some multidimensional generalization of this result.

Definition

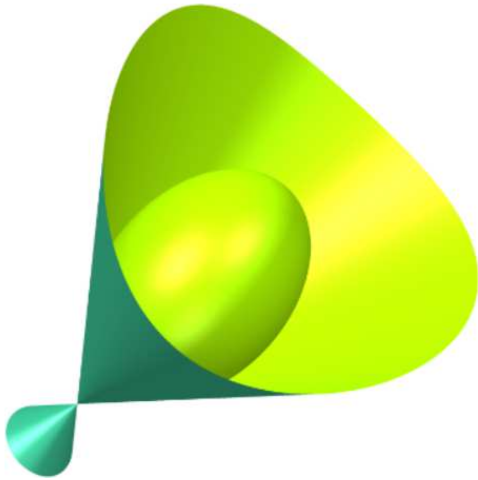
For $n \geq 2$, let us consider a square-free form $f(x_0, \ldots, x_n)$ of degree $d \geq 2$. Let us fix a point U with homogeneous coordinates $(u_0 : \ldots : u_n)$. Every straight line passing through the point U consists of points with homogeneous coordinates $((x_0 - u_0)t + u_0 s : \cdots : (x_n - u_n)t + u_n s)$, where $(s : t)$ are homogeneous coordinates inside the line. The restriction of the form f is a binary form denoted by $r(s, t)$.

Let us denote by $D[f, U]$ the discriminant of the binary form $r(s, t)$.

If $x_0 = 1$, then the discriminant is a inhomogeneous polynomial in affine coordinates x_k. In the general case, its degree is equal to $d^2 - d$.

If a straight line either is tangent to the hypersurface or passes through a singular point, then the discriminant of the form $r(t, s)$ vanishes. So, if the point U is not any singular point of the hypersurface, then the polynomial $D[f, U](x_1, \ldots, x_n)$ defines a cone with U as a vertex. If U is singular, then $D[f, U]$ vanishes identically.

Over the field of real numbers, a section of the cone that is defined by $D[f, U](x_1, \ldots, x_n) = 0$ is called the silhouette. Let us use the same notation over an arbitrary field.



In the general case, the polynomial $D[f, U](x_1, \ldots, x_n)$ is not homogeneous. So, we use affine space to define the silhouette.

One can define it in a projectively invariant way. A projective straight line passing through the point U corresponds to a two-dimensional linear space. So, the silhouette is naturally embedded into the Grassmannian.

The set of polynomials of the type $D[f,U]$ for all points U generates a linear subspace W_f of the ambient linear space of all inhomogeneous polynomials of degree d^2-d in n variables. The dimension of the ambient linear space is equal to

$$w(n,d) = \frac{(n+d^2-d)!}{n!(d^2-d)!}.$$

Theorem. *For every irreducible form f, the dimension* $\dim W_f$ *is projectively invariant.*

Remark. If a square-free form f is reducible, then any irreducible factor must not vanish at infinity.

Theorem. *If $d \geq 3$ and n is sufficiently large, then* $\dim W_f < w(n,d)$*, that is, W_f is a proper subspace of the ambient linear space.*

Theorem. *If the rank of a quadratic form f is equal to n, then the equality* $\dim W_f = w(n,2)$ *holds.*

Theorem (2017). *For given* n *and* d*, the dimension* $\dim W_f$ *considered as a function of coefficients of* f *is lower semi-continuous.*

Corollary. *If there exists a form* $f(x_0, \ldots, x_n)$ *of degree* d *such that* $\dim W_f = w(n, d)$*, then for almost every form* $f(x_0, \ldots, x_n)$ *of degree* d*, the equality* $\dim W_f = w(n, d)$ *holds too.*

Theorem (2017). *Given a square-free polynomial* $f(x_1, \ldots, x_n)$*. In the expansion of the polynomial* $D[f, U]$ *in powers of coordinates of the point* U*, each coefficient belongs to the linear subspace* W_f*. These polynomials in variables* $x_1, \ldots, x_n$ *span whole linear subspace* W_f*.*

Corollary. *There exists a polynomial time algorithm to compute the dimension of the linear subspace* W_f*.*

Corollary. *There exists a polynomial time algorithm to compute the dimension of the linear subspace* W_f*.*

It is sufficient to calculate the rank of a matrix whose order equals $w(n,d)$.
It requires $O(w^\omega)$ multiplications, where ω denotes the matrix multiplication exponent (Schönhage A. (1973) Unitäre Transformationen großer Matrizen, *Numerische Mathematik*, 20, 409–417).
In small dimensions, the rank can be calculated with computer algebra system software. With Maple, one can use $with(LinearAlgebra) : Rank()$.
With MathPartner, one can calculate the echelon form of a matrix.

For the manual how to compute $\dim W_f$, refer to:

Seliverstov A.V. (2017) On tangent lines to affine hypersurfaces [in Russian], *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, vol. 27, no. 2, pp. 248–256.
`https://doi.org/10.20537/vm170208`

Example

The vanishing locus of the polynomial $f = xy + 1$ is an affine hyperbola.

Let us denote affine coordinates of a point U by u and v. Expansion of the polynomial $D[f, U]$ is equal to

$$D[f, U] = y^2u^2 - (2xy + 4)uv + 4yu + x^2v^2 + 4xv - 4xy.$$

The linear subspace W_f coincides with the ambient space of all bivariate polynomials of degree at most two.

Its dimension is $\dim W_f = 6$. Its basis consists of polynomials:

$D_{uu} = y^2$, $D_{uv} = -2xy - 4$, $D_{vv} = x^2$, $D_u = 4y$, $D_v = 4x$, $D_1 = -4xy$.

We have computed $\dim W_f$ for certain **plane curves** ($n = 2$). In this case, the linear subspace W_f can be improper. But it is small for some curves.

d	2	3	4	5	6	7	8	9	10
$w(2,d)$	6	28	91	231	496	946	1653	2701	4186
$\dim W_{F_2}$	6	26	82	207	446	856	1506	2477	3862

where $F_2 = x_0^d + x_1^d + x_2^d$.

Theorem. *If $f(x_0, x_1, x_2)$ defines a singular plane curve, then the strict inequality* $\dim W_f < w(2,d)$ *holds.*

For almost every f of degree $d \leq 7$, the equality $\dim W_f = w(2,d)$ *holds.*

For all $d \leq 7$, the equality

$$\max_{f(x_0,x_1,x_2)} \dim W_f = w(2,d)$$

holds. In particular, the equality holds at forms of the type

$$f = x_0^d + x_1^d + x_2^d + (x_0 + x_1 + x_2)^d$$

We guess that it holds for every larger degree too. But it is hard to verify because, for d=7, the running time is approximately 11 hours.

Let us consider **cubic forms of the Fermat type**

$$F_n = x_0^3 + \cdots + x_n^3.$$

The polynomial $D[F_n, U](x_1, \ldots, x_n)$ is equal to the discriminant of a binary form of the type $at^3 + bt^2s + pts^2 + qs^3$ whose coefficients are sums of univariate polynomials, that is,

$$\begin{array}{rcl} a & = & a_1(x_1) + \cdots + a_n(x_n) \\ b & = & b_1(x_1) + \cdots + b_n(x_n) \\ p & = & p_0 + p_1x_1 + \cdots + p_nx_n \end{array}$$

and q is a constant. So, every monomial of

$$D[F_n, U] = b^2p^2 - 4ap^3 - 4b^3q - 27a^2q^2 + 18abpq$$

depends on at most four variables. Thus, $\dim W_{F_n} = O(n^4)$.

For $n \leq 9$, the equation $\dim W_{F_n} = \frac{1}{4}n^4 + \frac{5}{6}n^3 + \frac{9}{4}n^2 + \frac{8}{3}n + 1$ holds.

n	2	3	4	5	6	7	8	9
$\dim W_{F_n}$	26	72	165	331	602	1016	1617	2455

We have also computed $\dim W_f$ for certain **cubic hypersurfaces**.
For $n \leq 3$, this result found by symbolic computations with parameters, where every parameter can be considered as a transcendental number.
For $n \geq 4$, $\dim W_f$ was only computed for certain cubic forms. They provide the lower bound on the maximum value of $\dim W_f$ for given n.
For cubic forms $f(x_0, \ldots, x_n)$, we guess that the maximum dimension

$$\max_f \dim W_f = n + \dim W_{F_n} = \frac{1}{12}(n+1)(3n^3 + 7n^2 + 20n + 12)$$

n	2	3	4	5	6	7	8	9
$w(n,3)$	28	84	210	462	924	1716	3003	5005
$\max_f \dim W_f$	28	75	≥ 169	≥ 336	≥ 608	≥ 1023	≥ 1625	
$\dim W_{F_n}$	26	72	165	331	602	1016	1617	2455

Remark. For quadratic forms, the equalities $\max_f \dim W_f = \dim W_{F_n}$ hold. For quartic and quintic forms, the difference between $\max_f \dim W_f$ and $\dim W_{F_n}$ is larger than n.

Let us consider **cubic curves**. In the general case, $\dim W_f = 28$ except the Fermat type curves and all singular curves. We computed the determinant of a matrix composed by coefficients of polynomials generating the linear space W_f. For the Weierstrass normal form $f = x_2^2 x_0 + x_1^3 + p x_1 x_0^2 + q x_0^3$, the determinant is proportionate to the expression $p^4(4p^3 + 27q^2)^8$.
If $p = 0$ and $q \neq 0$, then the curve is projectively equivalent to a curve of the Fermat type. If $4p^3 + 27q^2 = 0$, then the curve is singular, else it is smooth.
For the Fermat cubic curve, $\dim W_{F_2} = 26$.
In this case, the Hessian curve is the union of three straight lines.

For an irreducible cubic curve with a node, $\dim W_f = 25$.
For a cubic curve with a cusp, $\dim W_f = 21$.

Therefore, *one can distinguish between nodal and cuspidal curves.*

The general cubic surface is projectively equivalent to a surface defined by $x_0^3+x_1^3+x_2^3+x_3^3+p_0x_1x_2x_3+p_1x_0x_2x_3+p_2x_0x_1x_3+p_3x_0x_1x_2$, where p_0, p_1, p_2, and p_3 are independent parameters.

Wakeford E.K. (1920) On canonical forms, *Proceedings of the London Mathematical Society* (2), vol. 18, no. 1, pp. 403–410.
Emch A. (1931) On a new normal form of the general cubic surface, *American Journal of Mathematics*, vol. 53, no. 4, pp. 902–910.

A **cyclic** cubic surface is projectively equivalent to a surface defined by a form of the type $x_0^3+x_1^3+x_2^3+x_3^3+px_1x_2x_3$, where p is a parameter.
For almost every value of p, the surface is smooth.
There exists a Galois cover of degree 3 over projective plane.
A smooth cubic surface is cyclic iff its Hessian surface contains a plane.
The Hessian surface of the Fermat cubic (when $p=0$) is simply the union of four planes since it has four cyclic structures.

Dolgachev I., Duncan A. (2019) Automorphisms of cubic surfaces in positive characteristic, *Izvestiya: Mathematics*, vol. 83, no. 3, pp. 424–500. `https://doi.org/10.1070/IM8803`

For the general **cubic surface**, $\dim W_f = 75$.
For the general **cyclic** cubic surface, $\dim W_f = 74$.
For **the Fermat** cubic surface, $\dim W_{F_3} = 72$.
So, *if the Hessian surface contains a plane, then* $\dim W_f$ *is small.*

But for some **singular** surfaces, the equality $\dim W_f = 75$ holds too.
For example, it holds for $f = x_0^3 + px_0^2x_1 + x_1^3 + x_0x_2^2 + (x_0^2 + x_1^2 + x_2^2)x_3$,
where p is transcendental; the point $(0 : 0 : 0 : 1)$ is singular.

Therefore, *the approach does not allow one to decide whether a given cubic surface is smooth*.

If $f = x_0^3 + px_0^2x_1 + x_1^3 + x_0x_2^2 + x_1x_2x_3$, where p is transcendental, then $\dim W_f = 73$; the point $(0 : 0 : 0 : 1)$ is singular too.
If $f = x_0^3 + px_0^2x_1 + x_1^3 + x_0x_2^2 + x_0^2x_3$, where p is transcendental, then $\dim W_f = 48$; the point $(0 : 0 : 0 : 1)$ is singular too.

If a **cubic hypersurface** is the projective closure of the graph of a polynomial, then it contains a singular point at infinity. The singular point is not any ordinary double point.

For cubic forms $f(x_0, \ldots, x_n)$, we guess that the maximum dimension

$$\max_f \dim W_f = n + \dim W_{F_n}$$

But for cubic forms of the type $g = f(x_0, \ldots, x_{n-1}) + x_0^2 x_n$, we guess

$$\max_g \dim W_g \leq n - 2 + \dim W_{F_n}$$

Let $H_2 = x_0^3 + x_1^3 + x_0^2 x_2$. For $n \geq 3$, let H_n denote the cubic form

$$x_0^3 + \cdots + x_{n-1}^3 + x_0 x_1 x_2 + x_1 x_2 x_3 + \cdots + x_{n-2} x_{n-1} x_0 + x_0^2 x_n$$

n	2	3	4	5	6	7	8
$\max_f \dim W_f$	28	75	≥ 169	≥ 336	≥ 608	≥ 1023	≥ 1625
$\dim W_{H_n}$	21	64	159	334	606	1021	1623

Conjecture. *For every cubic form g with reducible Hessian, the inequality holds*

$$\dim W_g < \max_f \dim W_f$$

Moreover, the more factors exists in Hessian, the more gap is between two values $\dim W_g$ *and* $\max_f \dim W_f$*.*

In particular, it holds for curves and surfaces as well as for all Fermat type hypersurfaces in sufficiently small dimensions.

For every cubic form of the type $g = f(x_0, \ldots, x_{n-1}) + x_0^2 x_n$, its Hessian is reducible too.

For **quartics**, these calculations are time-consuming.
If $n = 2$ or $n = 3$, then the linear subspace W_f can be improper. In these cases, if f defines a singular curve or surface, then the strict inequality $\dim W_f < w(n, 4)$ holds.
The table below contains $\dim W_{B_n}$ for hypersurfaces with 2^n singular points, where

$$B_n = \sum_{k=1}^{n} (x_k^2 - x_0^2)^2.$$

n	2	3	4
$w(n, 4)$	91	455	1820
$\max_f \dim W_f$	91	455	≥ 1792
$\dim W_{F_n}$	82	374	1325
$\dim W_{B_n}$	79	423	1740

For **quintics**, these calculations are time-consuming too.
If $n=2$, then the linear subspace W_f can be improper. So, if f defines a singular plane curve, then the strict inequality $\dim W_f < w(2,5)$ holds.

n	2	3
$w(n,5)$	231	1771
$\dim W_{F_n}$	207	1467

where $F_n = x_0^5 + \cdots + x_n^5$.

Plane sextic, septic, octic, nonic, and decic curves have been considered too, where $F_2 = x_0^d + x_1^d + x_2^d$.

d	2	3	4	5	6	7	8	9	10
$w(2,d)$	6	28	91	231	496	946	1653	2701	4186
$\dim W_{F_2}$	6	26	82	207	446	856	1506	2477	3862

Conclusion

The computational results show that one can easily verify smoothness of almost every plane quartic curve as well as almost every quartic surface in $\mathbb{P}^3$ by means of computing $\dim W_f$. The method is also applicable to other plane curves.
On the other hand, the same problem for cubic surfaces is hard enough because the proposed projective invariant is useless in this case.
Nevertheless, one can recognize singularities of some types.
We also assume that our method allows us to recognize cubic hypersurfaces with reducible Hessian in deterministic polynomial time.

The figures were made with the Surfer program
`http://www.imaginary.org/program/surfer`

Thank you!