Covariance of Parametric Representations of Orthogonal and Symplectic Matrices

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Abstract. Symplectic matrices are subject to certain conditions that are inherent to the Jacobian matrices of transformations preserving the Hamiltonian form of differential equations. A formula is derived which parameterizes symplectic matrices by symmetric matrices. An analogy is drawn between the obtained formula and the Cayley formula that connects orthogonal and antisymmetric matrices. It is shown that orthogonal and antisymmetric matrices are transformed by the covariant law when replacing the Cartesian coordinate system. Similarly, the covariance of transformations of symplectic and symmetric matrices is proved. From Cayley formulas and their analog, a series of matrix relations is obtained which connect orthogonal and symmetric matrices, together with similar relations connecting symplectic and symmetric matrices.

1. Cayley formulas. Covariance of parameterization of orthogonal matrices.

For orthogonal matrices, the Cayley formulas are known [1]

$$O=(E-K)(E+K)^{-1}, \quad K=(E-O)(E+O)^{-1} \tag{1}$$

They express an orthogonal matrix through an antisymmetric matrix $K^T=-K$.

Here E the unit (identical matrix), the upper index$T$ means the conjugation sign. For any antisymmetric matrix K, the matrix O satisfies the orthogonality conditions $OO^T=O^TO=E$. For any orthogonal matrix O (the eigenvalue is not -1), the matrix K satisfies the antisymmetry conditions.

The transition from one Cartesian system to another can be performed using an orthogonal matrix $C, \quad CC^T=E$. Let the matrix O determine the conversion of a radius vector $\mathbf{r}$ to a radius vector $\mathbf{R}:\mathbf{R}=O\mathbf{r}$. In the new coordinate system we have $\mathbf{r}'=C\,\mathbf{r}, \quad \mathbf{R}'=C\,\mathbf{R}$. Find the transformation matrix $\mathbf{R}'=O\mathbf{r}'$. Let's write this transformation in the original coordinate system $C\mathbf{R}=O'C\mathbf{r}\Rightarrow\mathbf{R}=$

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$C^T O' C\mathbf{r}$. Hence the law of transformation $O = C^T O' C \Rightarrow$

$$O' = COC^T \tag{2}$$

The corresponding transformation law for the matrix K in Cayley's formula follows

$$K' = CKC^T \tag{3}$$

Thus, the transformations of (2) and (3) matrices O and K are both covariant.

2. Covariance of parameterization of symplectic matrices.

A matrix $2n \times 2n$ is called symplectic if it satisfies the relation

$$AIA^T = I, \quad I = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}, \tag{4}$$

where E_n identity matrix $n \times n$.

Symplectic matrices are used in Hamiltonian mechanics. Such a matrix A is the Jacobian transformation matrix of a system of differential equations that preserves the Hamiltonian form [2, 3]. An analogy can be drawn between formulas (1) - (3) and the corresponding formulas for symplectic matrices.

To do this, we must match the orthogonal matrix O – the symplectic matrix and the antisymmetric matrix K to the product of the matrix I and the symmetric matrix $\frac{1}{2}\Psi$. Briefly this correspondence is written as follows $O \to A, \quad K \to \frac{1}{2}I\Psi$.

This analogy is shown in Table 1. On the left of the formula for O and K, on the right for A and $\frac{1}{2}I\Psi$.

$OO^T = O^TO = E \Rightarrow \det(O) = 1$	$A\,I\,A^T = A^T I\,A = I \Rightarrow \quad \det(A) = 1$
$K^T = -K$	$\Psi^T = \Psi$
$O = (E-K)(E+K)^{-1} =$ $= (E+K)^{-1}(E-K) \Rightarrow$ $\det(E+K) = \det(E-K)$	$A = \left(E + \frac{1}{2}I\Psi\right)\left(E - \frac{1}{2}I\Psi\right)^{-1} =$ $= \left(E - \frac{1}{2}I\Psi\right)^{-1}\left(E + \frac{1}{2}I\Psi\right) \Rightarrow$ $\det\left(E + \frac{1}{2}I\Psi\right) = \det\left(E - \frac{1}{2}I\Psi\right)$
$K = (E-O)(E+O)^{-1} =$ $= (E+O)^{-1}(E-O) \Rightarrow$ $\det(E-O)/\det(E+O) = \det(K)$	$\frac{1}{2}I\Psi = (E+A)^{-1}(A-E) =$ $= (A-E)(E+A)^{-1} \Rightarrow$ $\det(A-E)/\det(A+E) = \det\left(\frac{1}{2}I\Psi\right)$
$(E+K)\,(E+O) = 2E \Rightarrow$	$\left(E - \frac{1}{2}I\Psi\right)(E+A) = 2E \Rightarrow$
$\det\,(E \pm K)\det(E+O) = 2^{2n}$	$\det\left(E \pm \frac{1}{2}I\Psi\right)\det(E+A) = 2^{2n}$

TABLE 1. Analogy of parameterization of orthogonal and symplectic matrices

Table 2 in the left column shows the output of formulas for converting an orthogonal matrix O, when orthogonal coordinates are replaced with a matrix C. In the right column the same output of the symplectic matrix A transformation formulas for symplectic replacement of coordinates with the matrix B.

$C, \quad CC^T = E$	$B, \quad BIB^T = I$
$\mathbf{R} = O\mathbf{r}$	$\delta\mathbf{R} = A\delta\mathbf{r}$
$\mathbf{r}' = C\mathbf{r},\ \mathbf{R}' = C\mathbf{R}$	$\delta\mathbf{r}' = C\delta\mathbf{r},\ \delta\mathbf{R}' = C\delta\mathbf{R}$
$\mathbf{R}' = O'\mathbf{r}'$	$\delta\mathbf{R}' = A'\delta\mathbf{r}'$
$C\mathbf{R} = O'C\mathbf{r} \Rightarrow \mathbf{R} = C^TO'C\mathbf{r}$	$B\delta\mathbf{R} = O'B\delta\mathbf{r} \Rightarrow \delta\mathbf{R} = B^{-1}A'B\mathbf{r}$
$O = C^TO'C \Rightarrow O' = COC^T$	$A = B^{-1}A'B \Rightarrow A' = BAB^{-1}$

TABLE 2. Formulas for converting an orthogonal O and symplectic A matrices

$O' = COC^{-1}$	$A' = BAB^{-1}$
$O' = (E+K')^{-1}(E-K') =$ $= C\,(E-K)\,(E+K)^{-1}\,C^{-1}$	$A' = \left(E - \frac{1}{2}I\Psi'\right)^{-1}\left(E + \frac{1}{2}I\Psi'\right) =$ $=B\left(E + \frac{1}{2}I\Psi\right)\left(E - \frac{1}{2}I\Psi\right)^{-1}B^{-1}$
$(E-K')\,C\,(E+K) =$ $= (E+K')\,C\,(E-K)$	$\left(E + \frac{1}{2}I\Psi'\right)B\left(E - \frac{1}{2}I\Psi\right) =$ $\left(E - \frac{1}{2}I\Psi'\right)B\left(E + \frac{1}{2}I\Psi\right)$
$C\,(E+K-E+K) =$ $= K'C\,(E-K+E+K)$	$B\left(E - \frac{1}{2}I\Psi - E - \frac{1}{2}I\Psi\right) =$ $= \frac{1}{2}I\Psi'B\left(-E - \frac{1}{2}I\Psi - E + \frac{1}{2}I\Psi\right)$
$2CK = 2K'C \ \Rightarrow K' = CKC^T$	$BI\Psi = I\Psi'B \Rightarrow I\Psi' = BI\Psi B^{-1}$

TABLE 3. Proof of the covariance of the matrix transformation K and $(1/2)I\Psi$

Here $\mathbf{R} = O\mathbf{r}$ is an orthogonal conversion of the radius vector $\mathbf{r}$ to the radius vector $\mathbf{R}$ in the original coordinate system and $\mathbf{R}' = O'\mathbf{r}'$ the same conversion in the other coordinate system. Accordingly $\delta\mathbf{R} = A\delta\mathbf{r}$, $\delta\mathbf{R}' = A'\delta\mathbf{r}'$ are local symplectic transformations in the original and other coordinate systems.

Finally, Table 3 presents in the left column a proof of the covariance of the matrix transformation K that parameterizes the orthogonal transformation. In the right column, there is a similar proof of the covariance of the transformation of the matrix $(1/2)I\Psi$ that parameterizes the symplectic transformation.

Conclusion

Cayley formulas simplify the transformation of orthogonal matrices $O \to O'$. Let the matrix O be expressed as an antisymmetric matrix

$$K = \begin{pmatrix} 0 & k_3 & -k_2 \\ -k_3 & 0 & k_1 \\ k_2 & -k_1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

by Cayley formulas $O = (E-K)(E+K)^{-1}$. For orthogonal substitution with a matrix C , the conversion formula $O' = COC^T$ can be replaced with a vector transformation $\mathbf{k}' = C\mathbf{k}$, to express through vector $\mathbf{k}'$ antisymmetric matrix K' and by Cayley formulas find the transformed orthogonal matrix $O' = (E-K')(E+K')^{-1}$. Similarly, you can transform symplectic matrices $A = \left(E + \frac{1}{2}I\Psi\right)\left(E - \frac{1}{2}I\Psi\right)^{-1}$, expressing them through a symmetric matrix Ψ. Conversion formula $A' = BAB^{-1}$, where B - an arbitrary symplectic matrix can be replaced with the following sequence of transformations $I\Psi' = BI\Psi B^{-1}, \quad A' = \left(E + \frac{1}{2}I\Psi'\right)\left(E - \frac{1}{2}I\Psi'\right)^{-1}$.

Table 3 also implies identities of interest for determinants:

$$\det(E+K) = \det(E-K), \quad \det(E-O)/\det(E+O) = \det(K),$$
$$\det(E \pm K)\det(E+O) = 2^{2n}.$$

The corresponding formulas for symplectic matrices have the form

$$\det\left(E + \tfrac{1}{2}I\Psi'\right) = \det\left(E - \tfrac{1}{2}I\Psi'\right), \quad \det(A-E)/\det(A+E) = \det\left(\tfrac{1}{2}\Psi\right),$$
$$\det\left(E \pm \tfrac{1}{2}I\Psi\right)\det(E+A) = 2^{2n}.$$

In addition, for three-dimensional matrices, we have $\det(E \pm K) = 1 + \mathbf{k}^2$, $\det(E-O) = \det(K) = 0$.
From the last equality, it follows that the eigenvalue of an orthogonal matrix in three-dimensional space is 1 [2, 3].

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