Bar Code and involutiveness: Janet and Janet-like divisions

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Abstract. Involutive monomial divisions have been introduced by Janet as a concept [11, 12, 13, 14], being formally defined by Gerdt and Blinkov [6, 7]. In this talk we focus on two such particular divisions, namely Janet and Janet-like divisions [8, 9], treating them by means of Bar Codes, diagrams representing properties of monomial/semigroup ideals.

Extended abstract

Involutive monomial divisions have been introduced by Janet as a concept [11, 12, 13, 14], being formally defined by Gerdt and Blinkov [6, 7] and used to compute *involutive bases*, Groebner bases particularly efficient to find.

Consider the polynomial ring $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$ and the semigroup $\mathcal{T} \subset \mathcal{P}$ of terms in $x_1, ..., x_n$. Let $U \subset \mathcal{T}$ be a finite set of terms. For each $t \in U$, Janet defines a set $M(t, U) \subset \{x_1, ..., x_n\}$ of *multiplicative variables*[1] for t and calls *cone* of t the set $C(t, U) = \{t x_1^{\lambda_1} \cdots x_n^{\lambda_n} \mid \text{where } \lambda_j \neq 0 \text{ only if } x_j \in M(t, U)\}$; t is called *involutive divisor* for all the terms in $C(t, U)$ and only for them.

All cones are defined to be disjoint and Janet introduces a procedure, called *completion*, to enlarge U to a new set U' so that, called $\mathsf{T}(U)$ the semigroup generated by U,

$$\mathsf{T}(U) = \mathsf{T}(U') = \sqcup_{t \in U'} C(t, U').$$

If $U = U'$, then U is *complete*.

For an ideal $I = (f_1, ..., f_r) \triangleleft \mathcal{P}$, we consider the set of leading terms for its generators, namely $U = \{\mathsf{T}(f_1), ..., \mathsf{T}(f_r)\}$ and set Janet division on U, supposing U to be complete. A term $t \in \mathcal{T}$ is reducible by means of a generator f_i if and only if $t \in C(\mathsf{T}(f_i), U)$. This reduction procedure is that used to compute Janet involutive bases.

[1] The variables that are not multiplicative for t with respect to U form the set $NM(t, U)$ of non-multiplicative variables.

Janet-like division, defined in [8, 9], is a non-involutive generalization of Janet division, sharing many properties with the latter one. Instead of being based on the concept of multiplicative/non-multiplicative variables, it is based on non-multiplicative powers ($NM(t, U)$): the cone of a term t is given by the set of its multiples that are not divisible by a non-multiplicative power.

Bar Codes, introduced in [1, 2] are diagrams representing finite sets of terms and used in a series of papers in order to study the properties of monomial ideals. They are defined as follows.

Definition 0.1 **([1, 2]).** A Bar Code B is a picture composed by segments, called *bars*, superimposed in horizontal rows, which satisfies conditions $a., b.$ below. Denote by

- $\mathsf{B}_j^{(i)}$ the j-th bar (from left to right) of the i-th row (from top to bottom), $1 \leq i \leq n$, i.e. the *j-th i-bar*;
- $\mu(i)$ the number of bars of the i-th row
- $l_1(\mathsf{B}_j^{(1)}) := 1$, $\forall j \in \{1, 2, ..., \mu(1)\}$ the $(1-)$*length* of the 1-bars;
- $l_i(\mathsf{B}_j^{(k)})$, $2 \leq k \leq n$, $1 \leq i \leq k-1$, $1 \leq j \leq \mu(k)$ the *i-length* of $\mathsf{B}_j^{(k)}$, i.e. the number of i-bars lying over $\mathsf{B}_j^{(k)}$

a. $\forall i, j$, $1 \leq i \leq n-1$, $1 \leq j \leq \mu(i)$, $\exists ! \overline{j} \in \{1, ..., \mu(i+1)\}$ s.t. $\mathsf{B}_{\overline{j}}^{(i+1)}$ lies under $\mathsf{B}_j^{(i)}$

b. $\forall i_1,\, i_2 \in \{1, ..., n\}$, $\sum_{j_1=1}^{\mu(i_1)} l_1(\mathsf{B}_{j_1}^{(i_1)}) = \sum_{j_2=1}^{\mu(i_2)} l_1(\mathsf{B}_{j_2}^{(i_2)})$; we will then say that *all the rows have the same length.*

It is possible to associate to a finite set of terms a Bar Code and, on the other side, a finite set of terms to every Bar Code. The association is made so that the exponents of the terms are related to their position in the Bar Code.

In this talk, we will see how to study Janet division and Janet-like division by means of the Bar Code.
In particular, for Janet division, we will see how to compute multiplicative variables, find the involutive divisor of a term and detect whether a given set U is complete with respect to Janet division.

Suppose $x_1 < x_2 < ... < x_n$, let $U \subset \mathcal{T} \subset \mathbf{k}[x_1, ..., x_n]$ be a finite set of terms and B the associated Bar Code.
First perform the following three steps:

a) $\forall 1 \leq i \leq n$, put a star symbol $*$ on the right of t $\mathsf{B}_{\mu(i)}^{(i)}$;

b) $\forall 1 \leq i \leq n-1$, $\forall 1 \leq j \leq \mu(i) - 1$ let $\mathsf{B}_j^{(i)}$ and $\mathsf{B}_{j+1}^{(i)}$ be two consecutive bars not lying over the same $(i+1)$-bar; put a star symbol $*$ between these two bars.

Proposition 0.2. [3] *Let $U \subseteq \mathcal{T}$ be a finite set of terms and let us denote by B_U its Bar Code. For each $t \in U$ x_i, $1 \leq i \leq n$ is multiplicative for t if and only if the i-bar $\mathsf{B}_j^{(i)}$ of B_U, over which t lies, is followed by a star.*

Proposition 0.3. [5] *Let $U \subseteq \mathcal{T}$ be a finite set of terms and* B *be its Bar Code. Let $t \in U$, $x_i \in NM(t,U)$ and $\mathsf{B}_j^{(i)}$ the i-bar under t.*
Let $s \in U$; it holds $s \mid_J x_i t$ if and only if

1. $s \mid x_i t$
2. *s lies over $\mathsf{B}_{j+1}^{(i)}$ and*
3. *for each j' appearing with nonzero exponent in $\frac{x_i t}{s}$ there is a star after the j'-bar under s.*

Theorem 0.4. [5] *Let $U \subseteq \mathcal{T}$ be a finite set of terms and* B *be its Bar Code. Then U is a complete set if and only if for each $t \in U$ and each $x_i \in NM(t,U)$, called $\mathsf{B}_j^{(i)}$ the i-bar under t, there exists a term $s \in U$ satisfying conditions* 1, 2, 3 *of Proposition* 0.3.

The Bar Code equipped with stars, that we employ to compute multiplicative variables is a reformulation of Gerdt-Blinkov-Yanovich *Janet tree* [10], but in the (equivalent) presentation given by Seiler [15]. We will see more in detail the relation between the two diagrams.

For Janet-like division, we note that non-multiplicative powers are no more than powers of Janet non-multiplicative variables; analogously to Janet division we have

Proposition 0.5 ([4]). *Let $U \subseteq \mathcal{T}$ be a finite set of terms and let us denote by B_U its Bar Code. Let $t \in U$, x_i a Janet-nonmultiplicative variable, $\mathsf{B}_l^{(i)}$ the i-bar under t and t' any term over $\mathsf{B}_{l+1}^{(i)}$. Then for*

$$k_i = \deg_i(t') - \deg_i(t),$$

$x_i^{k_i}$ is a non-multiplicative power for t.

Theorem 0.6 ([4]). *Let $U \subset \mathcal{T}$ be a finite set of terms,* B *its Bar Code, $t \in U$, $p = x_i^{k_i} \in NMP(t,U)$ a nonmultiplicative power and $\mathsf{B}_j^{(i)}$ the i-bar under t. Let $s \in U$; $s \mid tp$ w.r.t. Janet-like division if and only if the following conditions hold:*

1. $s \mid pt$
2. *s lies over $\mathsf{B}_{j+1}^{(i)}$ and*
3. *$\forall j'$ such that $x_{j'} \mid \frac{pt}{s}$ either there is a star after the j'-bar under s or the nonmultiplicative power w.r.t. $x_{j'}$ has greater degree $deg_{j'}(\frac{pt}{s})$.*

Thanks to Theorem 0.6 completeness with respect to Janet-like division can be easily detected, since being complete only means that for each $t \in U$ and for each non-multiplicative power $p = x_i^{k_i}$ for t, there is a term s in U whose cone contains pt. Theorem 0.6 essentially says that we have to look for the term s in the next i-bar with respect that under t and check conditions 1,2,3.

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