Dynamic systems with quadratic integrals

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Abstract. In the report we discuss the problems of constructing difference schemes that mimic the properties of dynamic systems. We show how these problems can be solved in systems with quadratic integrals and how a many-body problem can be reduced to such systems.

One of the most widespread mathematical models is a dynamic system described by an autonomous system of ordinary differential equations, i.e., the system of the form

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_n), \quad i = 1, 2, \dots n, \tag{1}$$

where t is an independent variable, commonly interpreted as time, and the variables $x_1, \dots, x_n$ depending on it as coordinates of a point of several points. In applications the sight-hand sides f_i are often rational or algebraic functions of the coordinates $x_1, \dots, x_n$ or can be reduced to such form using a certain change of variables. As a rule, from physical reasons a few integrals of motion are known, but they are not sufficient to reduce the system of differential equations to Abel quadratures.

For example, the classical problem of n bodies [1] consists in finding solutions to the autonomous system of ordinary differential equations

$$m_i \ddot{\vec{r}}_i = \sum_{j=1}^{n} \gamma \frac{m_i m_j}{r_{ij}^3} \left(\vec{r}_j - \vec{r}_i\right), \quad i = 1, \dots, n \tag{2}$$

Here $\vec{r}_i$ is the radius vector of the i-th body and r_{ij} is the distance between the i-th and j-th body. This dynamic system is a Hamiltonian system of the order $2 \cdot 3 \cdot n$. For reducing it to quadratures using the Liouville method it is necessary to find $3 \cdot n$ algebraic integrals of motion in involution [2]. At the time of Liouville, only ten independent algebraic integrals of the many-body problem were known, which were called classical. In the 1880s, Bruns proved that every other algebraic integral of this problem is expressed in terms of these ten [2, 3]. This means that the many-body problem cannot be reduced to quadratures by the Liouville method. The question of whether it can be reduced to Abelian quadratures in another way was formulated by Bruns himself and resolved negatively [3, n. 23].

Classical explicit difference schemes, including explicit Runge-Kutta schemes, do not preserve these integrals. However, among implicit difference schemes there are schemes that preserve some classes of integrals of motion. The most studied are symplectic Runge-Kutta schemes that preserve all quadratic integrals of motion. For example, for a linear oscillator or a system of several coupled oscillators, these schemes allow organizing the calculation of the approximate solution in such a way that all the integrals of this system are preserved. In this case, the approximate solution mimics the periodicity of the exact solution, for example, you can choose a time step so that the approximate solution is a periodic sequence [4].

The construction of such mimetic schemes in the case of nonlinear dynamical systems is complicated by the appearance of non-quadratic integrals. For example, in the classical many-body problem by the Bruns theorem, there are 10 independent algebraic integrals, of which 9 are quadratic and therefore are preserved using any symplectic scheme. The first finite-difference scheme for the many-body problem, preserving all classical integrals of motion, was proposed in 1992 by Greenspan [5, 6] and independently in somewhat different form by J.C. Simo and O. González [7, 8]. The Greespan scheme is a kind of combination of the midpoint method and discrete gradient method.

In other site the standard symplectic schemes will preserve all integrals if we introduce the new variables such a way that all classical integrals are quadratic with respect of new variables. This approach is close to the invariant energy quadratization method (IEQ method) which was first proposed by Yang et al. [9] and used by Hong Zhang et al. [10] to conserve the energy at discretization of Hamiltonian systems including Kepler two-body problem. We applicate the same idea in many body problem.

First of all, we get rid of irrationality by introducing new variables r_{ij}, related to the coordinates by the equation

$$r_{ij}^2 - (x_i - x_j)^2 - (y_i - y_j)^2 - (z_i - z_j)^2 = 0$$

Then we eliminate the denominators in the energy integral by introducing new variables ρ_{ij}, related to the already introduced ones by the equations

$$r_{ij}\rho_{ij} = 1.$$

Note that this relation is quadratic again, so that after introducing additional variables this relation will turn into an additional quadratic integral.

For the sake of brevity let us denote the velocity components of the i-th body as $\dot{x}_i = u_i$, $\dot{y}_i = v_i$, and $\dot{z}_i = w$ and combine them into vector $\vec{v}_i$. From the many-body problem we pass to a system that consists of three coupled subsystems, namely, the system for coordinates

$$\dot{\vec{r}}_i = \vec{v}_i, \quad i = 1, \dots, n \tag{3}$$

the system for velocities

$$m_i \dot{\vec{v}}_i = \sum_{j=1}^{n} \gamma \frac{m_i m_j \rho_{ij}}{r_{ij}^2} \left(\vec{r}_j - \vec{r}_i\right), \quad i = 1, \dots, n \tag{4}$$

the system for distances

$$\dot{r}_{ij} = \frac{1}{r_{ij}}(\vec{r}_i - \vec{r}_j) \cdot (\vec{v}_i - \vec{v}_j), \quad i, j = 1, \dots, n;\ i \neq j. \tag{5}$$

and the system for inverse distances

$$\dot{\rho}_{ij} = -\frac{\rho_{ij}}{r_{ij}^2}(\vec{r}_i - \vec{r}_j) \cdot (\vec{v}_i - \vec{v}_j), \quad i, j = 1, \dots, n;\ i \neq j. \tag{6}$$

This system possesses 10 classical integrals of the many-body problem and additional integrals

$$r_{ij}^2 - (x_i - x_j)^2 - (y_i - y_j)^2 - (z_i - z_j)^2 = \text{const}, \quad i \neq j \tag{7}$$

and

$$r_{ij}\rho_{ij} = \text{const}, \quad i \neq j. \tag{8}$$

The autonomous system of differential equations (3)-(6), involving $n(n-1)$ additional variables r_{ij} and ρ_{ij}, has the following properties:

1. this system has quadratic integrals of motion (7) and (8), that allow expressing the additional variables r_{ij} and ρ_{ij} in terms of the coordinates of the bodies,
2. if the constants in these integrals are chosen such that
$$r_{ij}^2 - (x_i - x_j)^2 - (y_i - y_j)^2 - (z_i - z_j)^2 = 0 \quad \text{and} \quad r_{ij}\rho_{ij} = 1,$$
the solutions of the new system coincide with the solutions of the original one,
3. the new system has quadratic integrals of motion, which, with the relation between the additional variables and the coordinates of the bodies taken into account, turn into 10 classical integrals of the many-body problem.

Since all the classical integrals of the many-body problem, as well as the additional integrals in the new variables are quadratic, any symplectic Runge-Kutta difference scheme, including the simplest of them, the midpoint scheme, preserves all these integrals for sure.

Moreover, the autonomous system of differential equations (3)-(6) preserves the symmetry of the original problem with respect to permutations of bodies and time reversal, as, for example, the midpoint scheme.

At each step of the midpoint scheme, new values will be determined not only for the coordinates and velocities of the bodies, but also for auxiliary quantities r_{ij} and ρ_{ij}. If at the initial moment of time only the coordinates and velocities were specified, and the auxiliary variables were defined by equalities

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}, \quad \rho_{ij} = \frac{1}{r_{ij}},$$

then these equalities are preserved exactly (maybe up to the radical's signs) due to the fact that the auxiliary integrals (7) and (8) are quadratic and are preserved exactly when using the midpoint scheme; therefore, the quantities r_{ij} and ρ_{ij} do not lose their original meaning of the distances between the bodies and the inverse distances between the bodies.

Therefore, the midpoint scheme written for the system (3) - (6) preserves all its algebraic integrals exactly and is invariant under permutations of bodies and time reversal.

The report will present the results of numerical experiments with a midpoint scheme with an emphasis on its mimetic character, see also [11].

It should also be emphasized that the many-body problem has been reduced to the problem, all of whose integrals are quadratic. Another classical mechanical problem, the gyroscope rotation problem, has the same properties. The same problems arise when introducing classical transcendental functions elliptic and Abelian. Therefore, we intend to investigate in more detail dynamical systems with quadratic integrals.

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