Power geometry in solving system of nonlinear polynomial equations

Akhmadjon Soleev

Abstract. Here we present basic ideas and algorithms of Power Geometry and give a survey of some of its applications. We present a procedure enabling us to distinguish all branches of a space curve near the singular point and to compute parametric of them with any degree of accuracy. Here for a specific example we show how this algorithm works.

Introduction

Many problems in mathematics, physics, biology, economics and other sciences are reduced to nonlinear polynomial equations or to systems of such equations. The solutions of these equations and systems subdivide into regular and singular ones. Near a regular solution the implicit function theorem or its analogs are applicable, which gives a description of all neighboring solutions. Near a singular solution the implicit function theorem is inapplicable, and until recently there had been no general approach to analysis of solutions neighboring the singular one. Although different methods of such analysis were suggested for some special problems.
We offer an algorithm for calculating branches of nonlinear polynomial systems of equations based on Power Geometry [**?**, **?**, **?**]. Here we will consider only to compute local and asymptotic expansions of solutions to nonlinear equations of algebraic classes. As well as to systems of such equations.But it can also be extended to other classes of nonlinear equations for such as differential, functional, integral, integro-differential, and so on [**?**, **?**, **?**].

1. Ideas and algorithms

are common for all classes of equations. Computation of asymptotic expansions of solutions consists of 3 following steps (we describe them for one equation f=0).

1. Isolation of truncated equations $\hat{f}_j^{(d)} = 0$ by means of generalized faces of the convex polyhedron Γ(f) which is a generalization of the Newton polyhedron. The first term of the expansion of a solution to the initial equation $f = 0$ is a solution to the corresponding truncated equation $\hat{f}_j^{(d)} = 0$.
2. Finding solutions to a truncated equation $\hat{f}_j^{(d)} = 0$ which is quasi homogenous. Using power and logarithmic transformations of coordinates we can reduce the equation $\hat{f}_j^{(d)} = 0$ to such simple form that can be solved. Among the solutions found we must select appropriate ones which give the first terms of asymptotic expansions.
3. Computation of the tail of the asymptotic expansion. Each term in the expansion is a solution of a linear equation which can be written down and solved.

Elements of plane Power Geometry were proposed by Newton for algebraic equation (1670). Space Power Geometry for a nonlinear autonomous system of ODEs were proposed by Bruno (1962) [?].

2. System of algebraic equations [?, ?]

Let an algebraic curve F be defined in C^n by the system of polynomial equations

$$f_i(X) \stackrel{\text{def}}{=} \sum a_{iQ} X^Q = 0, \quad Q = (q_1, \dots, q_n) \in D_i, \quad i = 1, \dots, n-1, \tag{1}$$

where $D_i \stackrel{\text{def}}{=} D(f_i) = \{Q : a_{iQ} \neq 0\}$. Let $X = (x_1, \dots, x_n) = 0$ be a singular point of F, i.e. all $f_i(0) = 0$ and $\operatorname{rank}(\partial f_i / \partial x_j) < n - 1$ in $X = 0$. Then several branches of F pass through the $X = 0$. Each branch has its own local uniformization of the form

$$x_i = \sum_{k=1}^{\infty} b_{ik} t^{p_{ik}}, \quad i = 1, \dots, n \tag{2}$$

where exponents p_{ik} are integers, $0 > p_{ik} > p_{ik+1}$, and coefficients b_{ik} are complex numbers, series converge for large $|t|$, i.e. $X \to 0$ for $t \to \infty$. We propose an algorithm for finding any initial parts of the expansion (2) for all branches of F.

3. Objects and algorithms of Power Geometry

Let us consider the finite sum of monomials

$$f(X) = \sum a_Q X^Q \tag{3}$$

without similar terms and $a_Q \in C$.

The set $D = \{Q : a_Q \neq 0\}$ is called the <u>support</u> of f. We assume that $D \subset \mathbf{Z}^n$, and we enumerate all points of D as $Q_1, \dots, Q_l$. At first with a help of <u>Newton polyhedrons</u> and its <u>normal cones</u> of polynomials f_i, we find a list of <u>truncated systems</u> [?]

$$\widehat{f}_i(X) \stackrel{\text{def}}{=} \sum a_{iQ} X^Q = 0, \quad Q \in D_{ij}^{(d_i)}(f_i), \quad i = 1, \dots, n-1. \tag{4}$$

Each of them is the first approximation of (1). By the power transformation

$$y_i = x_1^{\alpha_{i1}} \dots x_n^{\alpha_{in}}, \quad i = 1, \dots, n \tag{5}$$

we reduce the number of variables in the truncated system(3). The power transformation (4) resolves (only partly) the singularity $X = 0$ of the system (1). In the transformed system (1), we find all points Y^0 corresponding to the point $X = 0$ of F. We translate each Y^0 into the origin and repeat the procedure described above. After a finite number of such steps, we come to a system having unique local branch and we uniforms it by means of the Implicit Function Theorem. Returning to the initial coordinate X by inverse transformations we receive the branch in the form (2). Analogously we uniforms all other branches of the curve F near the origin $X = 0$ and all branches going to infinity and real branches of a real curve as well.

4. Computation of branches of solutions of the specific system (1) consists of the following 8 stages:

1. For each coordinate singular point X^0 we do parallel-transfer $X - X^0$, write the system in the form (1) and make following stages for each such system separately. We shall describe them for system (1).

2. For each f_i compute the Newton polyhedron M_i, all its faces $\Gamma_{ij}^{(d_i)}$, and normal cones $N_{ij}^{(d_i)}$ and sets $D_{ij}^{(d_i)}$.

3. Find all nonempty intersections $N_{1j}^{(d_1)} \cap \dots \cap N_{n-1j}^{(d_{n-1})}$ with all $d_i > 0$ and, for each of them, write the corresponding truncated system (9).

4. For each such system (9), compute vectors T_i and the matrix α by Theorem 3.1 and make corresponding transformations of (1) and (3) into (17) and (16).

5. Find all roots of (4) and, by computation of the matrix $G = (\partial g_i / \partial y_j)$ separate simple roots Y^0 of (17).

6. By Implicit Function Theorem, compute an initial part of expansions for the branch corresponding to the simple root Y^0 of (17).

7. For each non-simple root Y^0 of (17), compute the new system (19) and repeat the procedure until a full isolation of all branches.

8. By inverse transformations, write all branches in initial coordinates X.

Stages 1–4 were programmed in PC. Stages 5, 6, and 8 are essentially nonlinear but can be done by standard programs.

Here we want to note that the complexity of the truncated system (9) is defined to be the $(n-1)$-dimensional Minkowski mixed volume of the corresponding parallel-transfer faces.

So we got the following result: If we perform calculations for 1-4 using this procedure , then at each step we find all the roots of the corresponding truncated system of equations , and find all the curves of the roots of the truncated system of equations, we obtain a local description of each component in the small neighborhood of the starting point $X = 0$, in the form of expansions (3).

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Akhmadjon Soleev
Samarkand State University, Uzbekistan
e-mail: asoleev@yandex.ru