Average number of solutions and mixed symplectic volume

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Abstract. The famous Koushnirenko-Bernstein theorem, also known as theorem BKK, asserts that the number of solutions of a polynomial system is equal to the mixed volume of Newton polyhedra of polynomials. This theorem creates the interaction platform for algebraic and convex geometries, which is useful in both directions. Here some statement is given that we view as a smooth version of the BKK theorem.

Introduction

Let X be an n-dimensional manifold, $V_1, \ldots, V_n$ be finite dimensional vector subspaces in $C^\infty(X, \mathbb{R})$, and let V_i^* be their dual vector spaces. We consider the systems of equations

$$f_1 - a_1 = \ldots = f_i - a_i = \ldots = f_n - a_n = 0, \qquad (1)$$

$f_i \in V_i$, $f_i \neq 0$, $a_i \in \mathbb{R}$. Let $H_i = \{v^* \in V_i^* \mid v^*(f_i) = a_i\}$ be an affine hyperplane in V_i^*, corresponding to equation $f_i - a_i = 0$, and $H = (H_1, \ldots, H_n)$ be a tuple of hyperplanes, corresponding to system (1). For a measure Ξ on the set of tuples $(H_1, \ldots, H_n)$, we define *the average number of solutions* as an integral of a number of zeroes of (1) with respect to Ξ.

We fix smooth Banach metrics in the spaces V_i. Further, we assume that the unit balls of these metrics are smooth and strictly convex bodies. We use these metrics for, firstly, to construct the measure Ξ and, secondly, to construct the Banach convex bodies $\mathcal{B}_i$ in X. Banach convex body or B-body in X is a collection $\mathcal{B} = \{\mathcal{B}(x) \subset T_x^* X\}$ of centrally symmetric convex bodies in the fibers of the cotangent bundle of X. We define the mixed symplectic volume of B-bodies and prove that the average number of zeroes is equal to the mixed symplectic volume of B-bodies $\mathcal{B}_1, \ldots, \mathcal{B}_n$.

Main theorems

We choose the measure Ξ on the space of systems (1) equal to a product of Crofton measures in spaces V_i^*. Recall that a translation invariant measure on the Grassmanian of affine hyperplanes in Banach space is called a Crofton measure, if a measure of a set of hyperplanes, crossing any segment, equals to its length. Under certain smoothness conditions there exists a unique such measure.

The symplectic volume of $\bigcup_{x\in X}\mathcal{B}(x)\subset T^*X$ we call the volume of Banach body $\mathcal{B}=\{\mathcal{B}(x)\}$. Using Minkowski sum and homotheties, we consider linear combinations of convex sets with non-negative coefficients. The linear combination of B-bodies is defined by $(\sum_i\lambda_i\mathcal{B}_i)(x)=\sum_i\lambda_i\mathcal{B}_i(x)$. The volume of a linear combination $\operatorname{vol}(\lambda_1\mathcal{B}_1+\ldots+\lambda_n\mathcal{B}_n)$ is a homogeneous polynomial of degree n in $\lambda_1,\ldots,\lambda_n$.

Definition 1. *The coefficient of polynomial* $\operatorname{vol}(\lambda_1\mathcal{B}_1+\ldots+\lambda_n\mathcal{B}_n)$ *at* $\lambda_1\cdot\ldots\cdot\lambda_n$ *divided by* $n!$ *is called the mixed volume of* B*-bodies and is denoted by* $\operatorname{vol}(\mathcal{B}_1,\ldots,\mathcal{B}_n)$.

B-body corresponding to Banach space V of smooth functions on X appears as follows. Let $B\subset V$ be a unit Banach ball. Define the mapping $\theta\colon X\to V^*$, as $\theta(x)\colon f\mapsto f(x)$. Let $d_x\theta$ be a differential of θ at $x\in X$, and let $d_x^*\theta\colon V\to T_x^*$ be an adjoint operator. So we get a Banach body $\mathcal{B}=\{\mathcal{B}(x)=d_x^*\theta(B)\}$.

Let $U\subset X$ be an open set with compact closure. Denote by $\mathfrak{M}(U)$ the average number of solutions of (1) in U. Let $\mathcal{B}_i$ be a B-body in X, corresponding to the space of functions V_i, and let $\mathcal{B}_i^U$ be a restriction of $\mathcal{B}_i$ to U.

Theorem 1.

$$\mathfrak{M}(U)=\frac{n!}{2^n}\operatorname{vol}(\mathcal{B}_1^U,\ldots,\mathcal{B}_n^U),$$

where $\operatorname{vol}(\mathcal{B}_1^U,\ldots,\mathcal{B}_n^U)$ *is a mixed volume of* B*-bodies.*

The proof of the theorem is based on the calculations in the ring of normal densities constructed in [AK18].

However, the Crofton measure in Banach space is quite exotic. For example, if the unit ball in Banach space is not a zonoid, then the Crofton measure is not everywhere positive; see [SC06, K20]. Recall that the zonotope is a polyhedron, represented as the Minkowski sum of segments, and the zonoid is a limit of a sequence of zonotopes converging with respect to the Hausdorff metric. If the unit ball of the Banach metric is a zonoid, then we call this metric a *zonoid metric*. All ellipsoids are zonoids and, respectively, Euclidean metrics are zonoid metrics.

The non-positivity of the measure Ξ reduces the validity of the notion of average number of solutions. For this reason, in order to avoid the non-positivity of Crofton meaure, we consider the averaging process under the general families of positive measures on the manifolds of affine hyperplanes.

Theorem 2. *Let* V *be a finite dimensional vector space, and let* μ *be a translation invariant countably additive smooth positive measure on the manifold of affine hyperplanes in* V^**. Then there is the unique Banach metric* $\|\,\|^\mu$ *in* V*, such that* μ *is a Crofton measure of the dual Banach metric in* V^**.*

Therefore, applying Theorem 1 for metrics corresponding to an arbitrary tuple of measures $\mu_1, \dots, \mu_n$ mentioned above, we obtain a version of the BKK *theorem for a general tuple of positive smooth measures.*

Remarks

Remark 1. *The metric* $\| \|^\mu$ *from Theorem* 2 *is a zonoid metric, and the corresponding* B*-body is a family of zonoids.*

Remark 2. *The case of Euclidean metrics in the spaces* V_i *was previously considered; see* [AK18, ZK14]. *In this case, the Banach bodies are ellipsoid families.*

Remark 3. *From Nash embedding theorem it follows that any smooth collection of ellipsoids in the fibers of* T^*X *can be obtained as* B*-body, corresponding to some Euclidean space of functions. If* X *is a compact manifold (may be with boundary), then from the Banach analogue of Nash theorem proved in* [BI94] *it follows that any collection* $\{\mathcal{B}(x)\colon\ x \in T^*_x X\}$ *of smooth strongly convex centrally symmetric bodies* $\mathcal{B}(x)$ *is a* B*-body, corresponding to some Banach space of functions on* X.

Remark 4. *Let* $\mu_1, \dots, \mu_n$ *be smooth translation invariant (not necessarily positive) measures on the manifolds of affine hyperplanes in* $V_1^*, \dots, V_n^*$ *respectively. Then the corresponding average number of solutions is equal to the mixed symplectic volume of some uniquely defined virtual Banach zonoids. Note that an arbitrary smooth centrally symmetric convex body is a virtual zonoid.*

References

[AK18] D. Akhiezer, B. Kazarnovskii, *Average number of zeros and mixed symplectic volume of Finsler sets*, Geom. Funct. Anal., vol. 28 (2018), pp.1517–1547.

[BI94] D. Yu. Burago, S. Ivanov, *Isometric embeddings of Finsler manifolds*, Algebra i Analiz 5 (1993), no. 1, 179 – 192 (Russian, with Russian summary); English transl., St. Petersburg Math. J. 5 (1994), no. 1, 159 – 169.

[K20] B. Kazarnovskii, *The average number of solutions.* To appear in Funct. Anal. Appl.

[SC06] Rolf Schneider, *Crofton measures in projective Finsler Spaces*, pp. 67–98, in Integral Geometry and Convexity, Wuhan, China, 18 – 23 October 2004, Edited By: Eric L Grinberg, Shougui Li, Gaoyong Zhang and Jiazu Zhou

[ZK14] D. Zaporozhez, Z. Kabluchko, *Random determinants, mixed volumes of ellipsoids, and zeros of Gaussian random fields*, Journal of Math. Sci., vol. 199, no.2 (2014), pp. 168–173.

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