Smooth version of BKK theorem

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October, 2017

Kushnirenko theorem

For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \subset \mathbb{R}^n$ we consider the function $x^m = x_1^{m_1} \ldots x_n^{m_n}$ called *the character of the complex torus* $(\mathbb{C} \setminus 0)^n$.

For a finite set $S \subset \mathbb{Z}^n$ the function $\sum_{m \in S, c_m \in \mathbb{C}} c_m x^m$ is called *Laurent polynomial with the support* S.

The space of Laurent polynomials with the support S we denote by V.

The convex hull $\Delta = \mathrm{conv}(S)$ we call the *Newton polyhedron* of the set S or the Newton polyhedron of the space V.

It follows from some general theorems of algebraic geometry that almost tuples of n functions from the space V have the same number of common zeros.
Denote this number by $\mathfrak{M}(V)$.

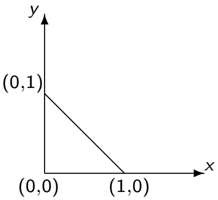
Kushnirenko theorem

$$\mathfrak{M}(V) = n! \, \mathrm{vol}(\Delta)$$

Example: linear equations

Linear equations

$n=2,\ S=\{(0,0),\ (1,0),\ (0,1)\},\ V=\{a+bx+cy\colon\ a,b,c\in\mathbb{C}\}$



$$2!\cdot\frac{1}{2}=1$$

What is a smooth version of Kushnirenko theorem?

Let X be an n-dimensional manifold (instead of the torus $(\mathbb{C} \setminus 0)^n$), and let $V \subset C^\infty(X, \mathbb{R})$ be a some finite-dimensional vector subspace (instead of the space of Laurent polynomials with a fixed support).
We consider the systems of equations

$$f_1 - a_1 = \ldots = f_i - a_i = \ldots = f_n - a_n = 0, \quad (1)$$

$0 \neq f_i \in V$, $a_i \in \mathbb{R}$.

The left-hand side of the equality of Kushnirenko theorem will be *the average number* $\mathfrak{M}(V)$ *of roots of system* (1) with respect to some measure on the space of systems.

The right-hand side will be *the volume of the Banach convex body in* X, corresponding to chosen measure.

The average number of roots

The choice of averaging measure is as follows. The Grassmanian of affine hyperplanes in the space E we denote by $\mathrm{AGr}^1(E)$. We identify the equation $f - a = 0$ with

$$H \subset V^* = \{v^* \in V^* \mid v^*(f) = a\} \in \mathrm{AGr}^1(V^*)$$

and respectively identify the system (1) with $(H_1, \ldots, H_n) \in \left(\mathrm{AGr}^1(V^*)\right)^n$.

Let μ_1 be a countably additive translation invariant smooth measure on $\mathrm{AGr}^1(V^*)$, and the measure μ_1^n be a corresponding measure on $\left(\mathrm{AGr}^1(V^*)\right)^n$. Thus the averaging measure $\Xi = \mu_1^n$ on the space of systems (1) obtained and the average number of roots $\mathfrak{M}(V)$ defined.

The average number of roots $\mathfrak{M}(V)$ depends on the choice of measure μ_1.

Banach bodies in X

Definition 1

Banach body (or B-body) in X is a collection of centrally symmetric convex bodies

$$\mathcal{B} = \{\mathcal{B}(x) \subset T_x^* X : x \in X\}$$

in the fibers of the cotangent bundle of X.

Definition 2

The volume of B-body $\mathrm{vol}(\mathcal{B})$ is defined as the volume of $\bigcup_{x \in X} \mathcal{B}(x) \subset T^* X$ with respect to the standard symplectic structure on the cotangent bundle. More precisely, if the symplectic form is ω then the volume form is $\omega^n / n!$.

Lemma

Introduce a Riemannian metric h on X. Let h_x be the corresponding metric on T_x and h_x^* the dual metric on T_x^*. Then $\mathrm{vol}(\mathcal{B}) = \int_X V(\mathcal{B}(x))dx$, where the volume $V(\mathcal{B}(x))$ of $\mathcal{B}(x)$ is measured with the help of h_x^*, and dx is the Riemannian n-density on X.

Banach bodies in X, II

Let V^* be a dual to V vector space. Define the mapping $\theta\colon X \to V^*$, as $\theta(x)(f) = f(x)$. Let $d\theta(x)\colon T_xX \to V^*$ be a differential of θ at x, and $d^*\theta(x)\colon V \to T^*_xX$ be an adjoint linear operator. For a centrally symmetric convex body $B \subset V$ define a B-body

$$\mathcal{B} = \{\mathcal{B}(x) = d^*\theta(x)(B) \subset T^*_xX\}.$$

From {D. Yu. Burago, S. Ivanov. Isometric embeddings of Finsler manifolds. St. Petersburg Math. J. 5 (1994), (5:1)} it follows that any smooth strongly convex B-body has a similar origin.

Let V be a Banach space, and B be a unit ball of Banach metric in V. We will say that the Banach body $\mathcal{B}$ defined above corresponds to the Banach space V.

Kushnirenko theorem formulation

Consider a Banach metric in V^*, dual to the metric of Banach space V. Recall that a translation invariant measure on $\mathrm{AGr}^1(V^*)$ is called *the Crofton measure*, if a measure of a set of hyperplanes, crossing any segment, equals to the length of this segment. It is true that *for a smooth Banach metric the Crofton measure exists and unique.*

Smooth Kushnirenko theorem

Let the unit ball of Banach metric in V is smooth and strongly convex. Then for average number of roots $\mathfrak{M}(V)$, measured with help of the Crofton measure in V^* is true that

$$\mathfrak{M}(V) = n! \cdot \mathrm{vol}(\mathcal{B}),$$

where $\mathcal{B}$ is the B-body corresponding to the Banach space V.

Mixed volume

Using Minkowski sum and homotheties, we consider linear combinations of B-bodies with non-negative coefficients

$$(\sum_i \lambda_i \mathcal{B}_i)(x) = \sum_i \lambda_i \mathcal{B}_i(x).$$

The volume $\mathrm{vol}(\lambda_1\mathcal{B}_1 + \ldots + \lambda_n\mathcal{B}_n)$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$.

Definition

The coefficient of polynomial $\mathrm{vol}(\lambda_1\mathcal{B}_1 + \ldots + \lambda_n\mathcal{B}_n)$ *at* $\lambda_1 \cdot \ldots \cdot \lambda_n$ *divided by* $n!$ *is called the mixed volume of B-bodies* $\mathcal{B}_1, \ldots, \mathcal{B}_n$ *and is denoted by* $\mathrm{vol}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$.

BKK theorem

Let $V_1, \dots, V_n$ be finite dimensional Banach spaces of $C^\infty(X, \mathbb{R})$-funstions, $\mu_i \in \mathrm{AGr}^1(V_i^*)$ be Crofton measures, $\Xi = \mu_1 \times \ldots \times \mu_n$ be a corresponding measure on the space $\mathrm{AGr}^1(V_1^*) \times \ldots \times \mathrm{AGr}^1(V_n^*)$ considering as a measure on the space of systems $f_1 - a_1 = \ldots = f_n - a_n = 0$, identified with tuples $(H_1, \dots, H_n) \in \mathrm{AGr}^1(V_1^*) \times \ldots \times \mathrm{AGr}^1(V_n^*)$. Set

$$\mathfrak{M}(V_1, \dots, V_n) = \int_{\mathrm{AGr}^1(V_1^*) \times \ldots \times \mathrm{AGr}^1(V_n^*)} N(H_1, \dots, H_n)\, d\Xi,$$

where $N(H_1, \dots, H_n)$ is a number of roots of corresponding system.

BKK theorem

$$\mathfrak{M}(V_1, \dots, V_n) = n! \cdot \mathrm{vol}(\mathcal{B}_1, \dots, \mathcal{B}_n),$$

where $\mathcal{B}_i$ is the B-body corresponding to the Banach space V_i.

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Hodge inequality, I

Let X be a homogeneous space of a compact Lie group. Assume that the spaces V_i are Euclidean and their scalar products are invariant. As a corollary from BKK theorem we get the Hodge inequalities similar to the well-known inequalities for intersection indices in algebraic geometry.

Theorem

$$\mathfrak{M}^2(V_1, \ldots, V_{n-1}, V_n) \geq \mathfrak{M}(V_1, \ldots, V_{n-1}, V_{n-1}) \cdot \mathfrak{M}(V_1, \ldots, V_n, V_n)$$

Proof.

All Banach ellipsoids $\mathcal{B}_i$ are invariant. Consequently

$$\mathrm{vol}(\mathcal{B}_1, \ldots, \mathcal{B}_n) = \mathrm{V}(\mathcal{B}_1(x), \ldots, \mathcal{B}_n(x)) \cdot \mathrm{vol}(X),$$

where $\mathrm{V}(\mathcal{B}_1(x), \ldots, \mathcal{B}_n(x))$ is a mixed volume, measured with respect to the metric of cotangent bundle, dual to invariant Riemannian metric in X. Therefore, the inequality is a consequence of the Alexandrov-Fenchel inequalities. □

Hodge inequality, II

Let $X = K/L$ be a Riemannian *isotropy irreducible homogeneous space*, Δ be the Laplace operator on X and $H(\lambda)$ be an eigenspace of Δ with eigenvalue λ, considered with L^2 metric. Put

$$\mathfrak{M}(\lambda_1, \dots, \lambda_n) = \mathfrak{M}(H(\lambda_1), \dots, H(\lambda_n)), \quad \mathfrak{M}(\lambda) = \mathfrak{M}(\lambda, \dots, \lambda).$$

It is known from {V.M.Gichev. Metric properties in the mean of polynomials on compact isotropy irreducible homogeneous spaces. Anal. Math. Phys. (2013), (3:2)} and {D.Akhiezer, B.Kazarnovskii. On common zeros of eigenfunctions of the Laplace operator. Abh. Math. Sem. Univ. Hamburg (2017), (87:1)} that

$$\mathfrak{M}(\lambda) = \frac{2}{\sigma_n n^{n/2}} \lambda^{n/2} \mathrm{vol}\,(X),$$

where σ_n is the volume of the n-dimensional sphere of radius 1.

For isotropy irreducible X the Hodge inequality becomes the equality.

Therefore (Gichev theorem)

$$\mathfrak{M}(\lambda_1, \dots, \lambda_n) = \frac{2}{\sigma_n n^{n/2}} \sqrt{\lambda_1 \cdot \dots \cdot \lambda_n}\, \mathrm{vol}\,(X).$$

Translation invariant measures and virtual Banach metrics

Let E be a Euclidean vector space. Denote by $\rho(H), \varphi(H)$ polar coordinates of the nearest to 0 point of $H \in \mathrm{AGr}^1(E)$. For any smooth even function $g\colon S \to \mathbb{R}$ the measure $g \cdot d\rho\, d\varphi$ is a smooth translation invariant measure on $\mathrm{AGr}^1(E)$. The space of such measures denote by $\mathfrak{m}_1(E)$.

Theorem

The space $\mathfrak{m}_1(E)$ does not depend on a choice of Euclidean metric in E.

Definition

Let the function $|x|\colon E \to \mathbb{R}$ be a difference of two Banach norms in vector space E. Then the function $|x|$ is called a virtual Banach metric.

Theorem

(1) For any $\mu \in \mathfrak{m}_1(E)$ there exists a unique smooth virtual Banach metric in E, such that μ is a Crofton measure for this virtual Banach metric.
(2) If the measure μ is positive then this virtual metric is real and zonoidal.
(3) On the contrary, for any zonoidal Banach metric positive Crofton measure exists and unique.

BKK theorem for virtual Banach metrics

Let $|x|$ be a virtual Banach metric in E, $|x| = |x|_1 - |x|_2$, where $|x|_1$, $|x|_2$ are Banach metrics in E, B_1, B_2 be the init balls of the dual Banach metrics in E^*. Then the virtual convex body $B_1 - B_2$ does not depend on the choice of $|x|_1$, $|x|_2$, and is called *the unit ball of the dual virtual metric*.

BKK for virtual Banach metrics

Let $\nu_1 \in \mathfrak{m}_1(V_1^*), \ldots, \nu_n \in \mathfrak{m}(V_n^*)$ be Crofton measures for virtual Banach metrics $|x|_i$ in the spaces V_i^*, $\mathfrak{M}(V_1, \ldots, V_n)$ be an average number of roots with respect to averaging measure $\nu_1 \times \ldots \times \nu_n$ on the space of systems. Then

$$\mathfrak{M}(V_1, \ldots, V_n) = \mathrm{vol}(\mathcal{B}_1, \ldots, \mathcal{B}_n),$$

where $\mathcal{B}_1, \ldots, \mathcal{B}_n$ are the virtual B-bodies corresponding to virtual unit balls in spaces $V_1, \ldots, V_n$.