Rigid body motion symmetries

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Abstract. The Galois axis, which constitutes an axis of "generalized" symmetry of a rigid body, is acted upon by the Klein four-group, generated by reflections across the principal axes, corresponding to extreme values of moments of inertia (minimal and maximal). In particular, the direction of the Galois axis might be reversed, prompting a remarkable duality of critical rigid body motion. Such reversal of direction of the Galois axis might be viewed as a composition of two distinct reflections, each of which would correspond to a time reversal symmetry.

The modular invariant as a symmetric function of the squares of the three principal moments of inertia

A torque free rigid body motion is governed by equations, possessing time and "mirror" symmetries. Several aspects of these symmetries were explored in [2, 3, 4, 5, 6, 12]. In [8], the (three) projections p, q, r of the angular velocity $\boldsymbol{\omega}$ upon the (three) principal axes of inertia, with corresponding moments of inertia A, B and C, are calculated via the Galois alternative elliptic function with corresponding elliptic moduli k_1, k_2 and k_3. The alternating group A_3 was shown to act on the squares k_1^2, k_2^2 and k_3^2 of the (three) elliptic moduli via the transformation $\tau : x \mapsto 1 - 1/x$, as further discussed in [9, 10]. Such a transformation (of order 3) would correspond to a cyclic permutation of the principal moments of inertia $(A\,B\,C)$, that is, putting

$$k^2(A, B, C) = \frac{(A - B)(Ch - m^2)}{(A - C)(Bh - m^2)},$$

with $k_1^2 = k^2(A, B, C)$, we must have $k_2^2 = k^2(B, C, A)$ and $k_3^2 = k^2(C, A, B)$, where the constants h and m^2 represent twice the kinetic energy and the square of the modulus of the angular momentum, respectively. The three elliptic moduli would, of course, correspond to one and the same value of the modular (Klein) invariant:

$$j = \frac{4(k^2+1/k^2-1)^3}{27(k^2+1/k^2-2)} = \frac{4(1-k_1^2-1/k_1^2)(1-k_2^2-1/k_2^2)(1-k_3^2-1/k_3^2)}{27} =$$
$$= 1 + \frac{4(k_1^2+k_2^2+k_3^2-3/2)^2}{27},$$

which is a symmetric function in the moments of inertia. It vanishes when k^2 coincides with a primitive cube root of -1, that is, a fixed point of the transformation τ. Another special value $j = 1$ is attained with

$$m^2 = \frac{C(A+B)-2AB}{2C-A-B}h, \;\; Bh - m^2 = \frac{(C-B)(B-A)h}{2C-A-B} \geqslant 0, \;\; A \leqslant B < C.$$

The rate of precession of a freely rotating rigid body as a symmetric function of its three principal moments of inertia

A formula for the rate of precession, about the (fixed) angular momentum $\boldsymbol{m}$, symmetric in the moments of inertia:

$$\dot{\psi} = \frac{1}{m}\left(h + \frac{(h-m^2/A)(h-m^2/B)(h-m^2/C)}{m^2\omega^2-h^2}\right),$$

where ω is the angular speed, was presented four years ago at the PCA 2016 conference [13]. The formula demonstrates that the rate of precession is uniform not only for permanent rotations about the axes, corresponding to extreme values of the moments of inertia, but it is also uniform for "permanent rotations" about the axis, corresponding to the intermediate value of the principal moments of inertia, which we label with the letter B. The last quote was taken to emphasize that the uniformity of the rate of precession does not necessarily imply a "usual permanent rotation" but it (the uniformity) holds, as well, for all (critical) solutions, satisfying $h = m^2/B$. Yet, the (improper) integral

$$\int_{-\infty}^{\infty}\left(\dot{\psi} - \frac{h}{m}\right)dt =: 2\theta(A,B,C),$$

taken with respect to time, throughout the critical motion does not vanish! For a triaxial rigid body the multivalued function $\theta(A,B,C)$ represents an angle between the Galois axis and a principal axis, corresponding to an extreme value of the moments of inertia. In fact,

$$\theta(A,B,C) = \sigma \operatorname{Arctan}\left(\sqrt{\frac{A(B-C)}{C(A-B)}}\right), \;\; \sigma = \begin{cases} 1, & \text{if } A<B<C, \\ -1, & \text{if } A>B>C,^1 \end{cases}$$

and the sum $\theta(A,B,C) + \theta(C,B,A)$ matches (modulo π) the angle $\pi/2$. A reflection of the Galois axis across a principal axis, corresponding to an extreme

[1]The sign flip reflects two distinct orientations of a coordinate system, fixed within a rigid body. In particular, an orientation of a coordinate system is changed with reversing the direction of the principal axis, corresponding to the intermediate moment of inertia.

moment of inertia, would rotate it (about the center of mass) by the angle 2θ, as calculated. Thus, such a reflection of the Galois axis corresponds not only to a time reversal but to a "flip" of the intermediate axis of inertia during the critical rigid body motion [11]. The said "mirror" reflection of the Galois axis might also be viewed as a reversal of a given orientation of a body-fixed coordinate system. A right-handed coordinate system is necessarily transformed into a left-handed system and vice versa. Such an observation is relevant to applications when the determination of the orientation must remain error free. A geometric interpretation of the "duality" of the Galois axis was well explained in [15], whereas an advice for correctly determining the (multivalued) angle θ was suggested by E. Mityushov in an email message sent to the author on February 27, 2018. It is based on an observation, concerning the Galois modulus

$$G(A, B, C) := \frac{C(B - A)}{B(C - A)},$$

which, upon ascendingly ordering the principal moments of inertia $A \leqslant B < C$, coincides with the square of the scalar product of two unit vectors, directed (respectively) along the Galois axis and the principal axis, corresponding to the minimal value of inertia (labeled with the letter A). The (ascending) ordering condition turns out being superfluous if one adopts an "invariant" way for defining the Galois modulus as a square of a scalar product of the said unit vectors, where the second is now aligned along a principal axis, corresponding to an extreme value of the moments of inertia, which need not necessarily be minimal. With this simplified (and thus improved) definition, the Galois modulus is seen to be the square of the scalar product of a unit vector along the Galois axis with a unit vector along a principal axis, corresponding to an extreme moment of inertia which we might still label with the letter A (but without further requiring A to be minimal).[2]

As emphasized in [7], the MacCullagh ellipsoid of inertia would "degenerate" to an ellipsoid of revolution whenever two of the principal moments would coincide one with the other. The Galois axis, being an axis orthogonal to the circular sections of such an ellipsoid would then coincide with the axis of (dynamical) symmetry.

Conclusion

The (tip of the) angular momentum pseudovector $\boldsymbol{m}$ might be viewed as a holomorphic (orientation preserving) function, mapping time to a fixed within a rigid body (unit) sphere. We must furthermore distinguish a "right" pseudovector, which coordinates are given with respect to a right-handed body-fixed coordinate system from its "mirror" reflection, that is, a "left" pseudovector which coordinates are given with respect to a (reflected) left-handed body-fixed coordinate system. Establishing such a distinction would

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[2]A clarifying formula would be $G(A, B, C) + G(C, B, A) = 1$.

protect us from a conventional (yet erroneous) reversal of the direction of the reflected pseudovector. In other words, a given “parity” of a pseudovector (right or left) cannot be altered by reversing its (preserved) direction, so we risk no confusion since we never alter a given pseudovector aside from subjecting it to transfomations which might (or might not) preserve its parity. Then, the pseudovector, obtained by so reflecting the initial (right) pseudovector $\boldsymbol{m}$ might be regarded as an antiholomorphic (orientation reversing) map. Ignoring such an elementary observation has apparently precluded the practical implementation of exact rigid body motion solution, commonly substituting it with numerical approximations. The duality rather than uniqueness of rigid body critical motion had prompted D. Abrarov to declare, in [1], “the Dzhanibekov's top” as the “classical analogue” of the sought after (in quantum field theory) “massless particle, possessing spin 2”, that is, “the graviton”! Dzhanibekov's top appears on a diagonal of the animation in [14]. A “dual top”, corresponding to an opposite (initial) rotation appears on the other diagonal. Each of the Galois axes, corresponding to these dual tops are reflections, one of the other across a “mirror”, orthogonal to a principal axis, corresponding to one of the two extreme values of the moments of inertia.

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