Study of the Liénard Equation by Means of the Method of Normal Form

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Abstract. The purpose of this report is the demonstration of searching the first integral of motion by the method of normal form. For the object of this demonstration, we chose the Liénard equation. We represented the equation as a dynamical system and parameterized it. After a calculation of the normal forms near stationary points, we found parameter values at which the condition of local integrability is satisfied for all stationary points simultaneously. We found two such sets of parameters. For each of them, the global integrability takes place.

Introduction

We use the approach based on the local analysis. It uses the resonance normal form calculated near stationary points [1]. In the paper [2] it was suggested the method for searching the values of parameters at which the dynamical system is locally integrable in all stationary points simultaneously. Satisfying the such local integrability conditions is a necessary condition of a global integrability. For global integrability of autonomous planar system, it is enough to have one global integral of motion. From its expression, you can get the solution of the system in quadratures. That is the integrability always leads to solvability. Note, that the converse is not true. Note also that a record of solutions via corresponding integrals is often more compact than an expression of the solution itself.

Problem

We will check our method on the example of the Liénard equation [3]

$$\ddot{x} = f(x)\dot{x} + g(x) = 0, \tag{1}$$

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which can be rewritten as a dynamical system. Let $f(x)$ be a quadratic polynomial and $g(x)$ is polynomial of fourth order. Then equation (1) is equivalent to the system

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= \left[a_0(t) + a_1 x(t) + a_2 x^2(t)\right] y(t) + b_1 x(t) + b_2 x^2(t) + b_3 x^3(t) + b_4 x^4(t), \end{aligned} \tag{2}$$

here parameters $a_0, a_1, a_2, b_1, b_2, b_3, b_4 \in \mathbb{R}, b_1 \neq 0$.

Method

The main idea of the discussed method is a search of conditions on the system parameters at which this system is locally integrable near its stationary points. The local integrability means we have enough number (one here) of the local integrals which are meromorphic near each stationary point. Local integrals can be different for each such point, but for the existence of the global integral, the local integrals should exist in all stationary points. This is a necessary condition. We have an algebraic condition for local integrability. It is the condition **A** [1, 2]. We look for sets of parameters at which the condition **A** is satisfied at all stationary points simultaneously. Such sets of parameters are good candidates for the existence of global integrals. These integrals we look for by other methods.

Condition of Local Integrability

The condition **A** is some infinite sequence of polynomial equations in coefficients of the system. Near each of the stationary points is its own equations. The condition of global integrability is a unification of these infinite systems of polynomial equations. Any part of this infinite system is a necessary condition of the integrability. We solve a finite subset of these equations. The condition **A** is formulated in terms of the normal form of the system. We calculated the resonance normal form for the system (2) near the stationary point in the origin using the MATHEMATICA 11 system and the program [4] till the 8^{th} order. After that, we wrote down the lowest equations (till the order of the eight) of the local integrability condition **A**. We solved this finite subsystem and got three sets of parameters.

1. $a_0 = 0, a_1 = 0, a_2 = 0$;
2. $a_0 = 0, a_1 b_2 = a_2 b_1, b_3 = 0, b_4 = 0$;
3. $a_0 = 0, a_2 = 0, b_2 = 0, b_4 = 0$.

Then we checked the condition of local integrability **A** near other stationary points. The third set above does not satisfy the local condition near some of the stationary points, so this is not a candidate for the global integrability.

First Integrals of Motion

For searching for global integrals, we divided the left and right sides of equations (2) into each other for each from the sets above. In result we had the first-order differential equations for $x(y)$ or $y(x)$. Then we solved them by the MATHEMATICA solver and got cumbersome solutions $y(x)$. After that we calculated the integrals from these solutions extracting the integration constants $I(x(t), y(t)) = const$. For the first set of parameters above, we got

$$I_1(x(t), y(t)) = 30\, y(t)^2 - 30\, b_1 x^2(t) - 20\, b_2 x^3(t) - 15\, b_3 x^4(t) - 12\, b_4 x^5(t). \quad (3)$$

Its time derivative $dI_1(t)/dt = 0$ along the system (2) over all phase space. So, it is the first integral.

For the second set we got

$$I_2(x(t), y(t)) = 6\, b_1^2 \, \log[b_1 + a_1 y(t)] - 6\, a_1 b_1 y(t) + a_1^2\, x^2(t)\, [3\, b_1 + 2\, b_2 x(t)], \quad b_1 \neq 0. \quad (4)$$

The time derivative $dI_2(t)/dt = 0$ along the system (2). The limitation on the positivity of the argument of the logarithm can be eliminated by the representation of integral I_2 in the form $I = \exp(I_2)$. Of course, later additional studying analytical properties of this integral and the phase picture of the system should be carried . Nevertheless, we have here the first integral.

Scheme

The proposed method is intended for a search of the values of parameters at which some a polynomial autonomous dynamical system is integrable. The main steps of the method are:

- calculation of the normal form at stationary points of the system ;
- calculation finite subsets of equations of **A** condition at these points;
- the solution of these subsets of the equations in system parameters at the one (or more) stationary points;
- verifying the fulfillment of the condition **A** for the found parameter sets at the rest stationary points.
- the found parameter sets are used for searching integrals of the system by other methods.

Of course, there are different tactics. For example, in the current paper, we solved the set of equations near the origin, then tried found integrals, and only after that checked the condition **A** at other stationary points. Our practice work with planar autonomous systems demonstrates that at such values of parameters corresponding integrals exist for most sets of solutions of **A**.

Note, the proposed method of searching suitable parameters has no limitation on a system dimension. The limitations arise on the stage of searching integrals of motion.

Result

We found integrability at two sets of parameters. Set (1) corresponds to equation (1) in the form

$$\ddot{x} = b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4.$$

The corresponding integral is (3).
Set (2) corresponds to equation (1) in the form

$$\begin{aligned}
&\ddot{x} = x\,(b_1 + b_2 x)\,(1 + a\,x),\\
&\text{or}\\
&\ddot{x} = b_1 x + b_2 x^2 + a\,(b_1 x + b_2 x^2)\,y,\\
&a \equiv a_1/b_1, \quad b_1 \neq 0.
\end{aligned}$$

The corresponding integral is (4).

Conclusion

We represented the Liénard equation as a dynamical system and parameterized it in a polynomial form. For this system, we found two sets of parameters at which it has the first integrals of motion and solvable. Both cases are trivial from a point of view of studying of Liénard's equation (1). The first case corresponds to equality $f(x) = 0$, the second one to $f(x) \sim g(x)$. But the workability of the method was illustrated.

The proposed method of searching suitable parameters has no limitation on a system dimension. The limitations arise on the stage of searching integrals of motion.

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