Study of the Liénard Equation by the Method of Normal Form

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Integrability

For an ODE system the integrals of motion satisfy the relation

$$\frac{d\ \ I_k(x_1,\ldots,x_n,t)}{d\ \ t} = 0 \quad \text{along the system} \quad \frac{d\ \ x_i}{d\ \ t} = \phi_i(x_1,\ldots,x_n,t),$$

$$i = 1,\ldots,n, \quad k = 1,\ldots,m$$

The integrals should satisfy also some additional properties. System is called integrable if it has enough numbers of the integrals. For integrability of an autonomous plane system, it is enough to have a single integral.

- Integrability is a very important property of the system. In particular, if a system is integrable then it is solvable by quadrature (but not vise versa).
- The knowledge of the integrals is important at investigation of a phase portrait, for a creation of symplectic integration schemes e.t.c.

Task

- Generally, the integrability is a rare property.
- But the system may depend on parameters.
- The task is to find the values of these parameters at which the system is integrable.

Example

For example, we have had treated the system

$$dx/dt = -y^3 - b\,x^3 y + a_0\,x^5 + a_1\,x^2 y^2,$$
$$dy/dt = (1/b)\,x^2 y^2 + x^5 + b_0\,x^4 y + b_1\,x\,y^3,$$

with five arbitrary real parameters $b \neq 0, a_1, a_2, b_1, b_2$.

Using the Power Geometry method, we brought the system to a non-degenerate form.

With our technique we found seven two-dimensional conditions at which the system above is integrable

1) $b_1 = 0,$ $\quad a_0 = 0, a_1 = -b_0\, b, \;\; b^2 \neq 2/3;$

2) $b_1 = -2\,a_1,$ $\quad a_0 = a_1\, b, b_0 = b_1\, b, \;\; b^2 \neq 2/3;$

3) $b_1 = 3/2\,a_1,$ $\quad a_0 = a_1\, b, b_0 = b_1\, b, \;\; b^2 \neq 2/3;$

4) $b_1 = 8/3\,a_1,$ $\quad a_0 = a_1\, b, b_0 = b_1\, b, \;\; b^2 \neq 2/3;$

5) $b_1 = 3/2\,a_1,$ $\quad a_0 = (2b_0 + b\,(3a_1 - 2b_1))/3, \;\; b = \sqrt{2/3};$

6) $b_1 = 6\,a_1 + 2\sqrt{6}b_0,$ $\quad a_0 = (2b_0 + b\,(3a_1 - 2b_1))/3, \;\; b = \sqrt{2/3};$

7) $b_1 = -2/3\,a_1,$ $\quad 20a_0 + 2\sqrt{6}a_1 + 4b_0 + 3\sqrt{6}b_1 = 0,$
$\quad 3a_0 - 2b_0 \neq b\,(3a_1 - 2b_1), \qquad b = \sqrt{2/3}.$

For each of these conditions we found the first integral of motion.

- In this example, we first found those 7 relations which were perspective for integrability, and then the integrals were found by some other methods.
- This example is sophisticate and for a demonstration of our technique, we need to consider a simpler sample.

Global & Local

- Let us see the plane autonomous system in the variables $\{x(t), y(t)\}$.
- We look for the global integral a function whose a full derivation in time equal zero along the system in some domain in the variables $\{x, y\}$.
- If we assume that this function is a single-valued function, then it must satisfy this condition at every point of the domain.
- Thus, the condition local of integrability at every point will be a necessary condition for global integrability.

Idea of the Technique

- The local analysis of ODEs is well developed.
- Generally, the global integral does not exist. But it may exist at some values of the system parameters.
- We solve the necessary conditions of the local integrability in the parametric space. We get sets of parameter values that are good candidates for a search of the global integrability.
- At these values of parameters, we will search the global integrals later.

Condition of the Local Integrability

- The resonance normal form was introduced by Poincaré for investigation of systems of nonlinear ordinary differential equations. It is based on the maximal simplification of the right-hand sides of these equations by invertible transformations.
- The normal form approach was developed in works of G.D. Birkhoff, T.M. Cherry, A. Deprit, F.G. Gustavson, C.L. Siegel, J. Moser, A.D. Bruno et.al. This technique is based on the Local Analysis method by Prof. Bruno.

Multi-index notation

Let's suppose that we treat the polynomial system and rewrite this n-dimension system in the terms

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{\mathbf{q}\in\mathbb{N}_i} f_{i,\mathbf{q}} \mathbf{y}^{\mathbf{q}}, \quad i = 1, \dots, n, \tag{1}$$

where we use the multi-index notation

$$\mathbf{x}^{\mathbf{q}} \equiv \prod_{j=1}^{n} x_j^{q_j},$$

with the power exponent vector $\mathbf{q} = (q_1, \dots, q_n)$
Here the sets:

$$\mathbb{N}_i = \{\mathbf{q} \in \mathbb{Z}^n : q_i \geq -1 \text{ and } q_j \geq 0\,,\ \text{if } j \neq i\,, \quad j = 1, \dots, n\}\,,$$

because the factor y_i has been moved out of the sum in (1).

Normal form

The normalization is done with a near-identity transformation:

$$x_i \;=\; z_i + z_i \sum_{\mathbf{q}\in\mathbb{N}_i} h_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1,\dots,n \tag{2}$$

after which we have system (1) in the normal form:

$$\dot{z}_i = \lambda_i z_i + z_i \sum_{\substack{\langle \mathbf{q},\mathbf{L}\rangle = 0 \\ \mathbf{q}\in\mathbb{N}_i}} g_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1,\dots,n\,. \tag{3}$$

Resonance terms

- The important difference between (1) and (3) is a restriction on the range of the summation, which is defined by the equation:

$$\langle \mathbf{q}, \mathbf{L} \rangle = \sum_{j=1}^{n} q_j \lambda_j = 0 \,. \tag{4}$$

- **L** here is the vector of the eigenvalues of matrix of linear part of the system (1). The **q**-terms in the normal form (3) are terms, for which (4) is valid. They are called resonance terms.

Note, if the eigenvalues are not comparable then condition (4) is never valid at any components of the vector **q**, because they are integer. For example such situation takes place if $\lambda_1 = 1, \lambda_2 = \sqrt{2}$. In that case the normal form (3) will be a linear system.

Calculation of the Normal form

The *h* and *g* coefficients in (2) and (3) are found by using the recurrence formula:

$$g_{i,\mathbf{q}}+\langle\mathbf{q},\mathbf{L}\rangle\cdot h_{i,\mathbf{q}} = -\sum_{j=1}^{n}\sum_{\substack{\mathbf{p}+\mathbf{r}=\mathbf{q}\\ \mathbf{p},\mathbf{r}\in\bigcup_i\mathbb{N}_i\\ \mathbf{q}\in\mathbb{N}_i}}(p_j+\delta_{ij})\cdot h_{i,\mathbf{p}}\cdot g_{j,\mathbf{r}}+\tilde{\Phi}_{i,\mathbf{q}}\,, \quad (5)$$

For this calculation we wrote two programs, in LISP and in high level language of the MATHEMATICA system.

Conditions A and ω

There are two conditions:

- Condition A. In the normal form (3)

$$g_j(Z) = \lambda_j a(Z) + \bar{\lambda}_j b(Z), \quad j = 1, \ldots, n,$$

  where $a(Z)$ and $b(Z)$ are some formal power series.
- Condition ω (on small divisors). It is fulfilled for almost all vectors **L**. At least it is satisfied at rational eigenvalues.
- If these conditions are satisfied then the normalizing transformation (2) converges.

Near a stationary point the condition **A:**

- Ensures convergence.
- Provides the local integrability.
- Isolates the periodic orbits if eigenvalues are pure imagine.

Near a stationary point the condition **A**

- The condition **A is an infinite system of algebraic equations on the system parameters.**
- **It can be calculated by the CA program till some finite order. It will be the necessary condition of the local integrability as a finite system of algebraic equations in the system parameters. It is solvable.**

The Liénard equation

- Let f and g be two continuously differentiable functions on $\mathbb{R}$, where g is an odd and f is an even function. Then the second order ordinary differential equation of the form

$$\ddot{x} = f(x)\dot{x} + g(x) = 0$$

  is called the Liénard equation.
- **Theorem** Liénard equation has a unique and stable limit cycle surrounding the origin if it satisfies the following additional properties...
- We will look at the case of arbitrary functions f and g.

Parametrization

- We choose f and g as the polynomials

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= \left[a_0(t) + a_1 x(t) + a_2 x^2(t)\right] y(t) + \\ &\quad + b_1 x(t) + b_2 x^2(t) + b_3 x^3(t) + b_4 x^4(t), \end{aligned}$$

  with real parameters $a_0, a_1, a_2, b_1, b_2, b_3, b_4$ and $b_1 \neq 0$.
- The origin is the stationary point here.

Solution of the Condition **A**

The condition **A** is formulated in terms of the normal form of the system. We calculated the resonance normal form for the Liénard system near the stationary point in the origin till the 8^{th} order, wrote down the lowest equations (till the order of the eight) of the local integrability condition **A**. We solved this finite subsystem and got three sets of parameters.

1. $a_0 = 0, a_1 = 0, a_2 = 0;$
2. $a_0 = 0, a_1 b_2 = a_2 b_1, b_3 = 0, b_4 = 0;$
3. $a_0 = 0, a_2 = 0, b_2 = 0, b_4 = 0.$

Then we checked the condition of local integrability **A** near other stationary points. The third set above does not satisfy the local condition near the non-origin stationary points, so this is not a candidate for the global integrability.

First Integrals of Motion

For searching for global integrals, we use the Lagutinsky metod. We divided the left and right sides of the system equations into each other.

$$\frac{dx(t)}{dt} = P(x(t), y(t)),$$

$$\frac{dy(t)}{dt} = Q(x(t), y(t)).$$

In result we have the first-order differential equations for $x(y)$ or $y(x)$

$$\frac{dx}{dy} = \frac{P(x, y)}{Q(x, y)} \quad \text{or} \quad \frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}.$$

Then we solved them by the MATHEMATICA solver and got cumbersome solutions $y(x)$. After that we calculated the integrals from these solutions by extracting the integration constants $I(x(t), y(t)) = const(t)$.

For the first set of parameters above, we got

$$I_1(x(t),y(t)) = \begin{array}{l} 30\,y(t)^2 - 30\,b_1 x^2(t) - 20\,b_2 x^3(t) - 15\,b_3 x^4(t) - \\ -12\,b_4 x^5(t). \end{array}$$

Its time derivative $dI_1(t)/dt = 0$ along the system over all phase space. So, it is the first integral. For the second set we got

$$I_2(x(t),y(t)) = \begin{array}{l} 6\,b_1^2\,\log[b_1 + a_1 y(t)] - 6\,a_1 b_1 y(t) + \\ +a_1^2\,x^2(t)\,[3\,b_1 + 2\,b_2 x(t)], \end{array}$$
$$b_1 \neq 0.$$

The limitation on the positivity of the argument of the logarithm can be eliminated by the representation of integral I_2 in the form $I = \exp(I_2)$. Of course, later additional studying analytical properties of this integral and the phase picture of the system should be carried . Nevertheless, we have here the first integral.

Results

We found integrability at two sets of parameters. Set (1) corresponds to the equation in the form

$$\ddot{x} = b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 .$$

Set (2) corresponds to the equation in the form

$$\begin{aligned} &\ddot{x} = x\,(b_1 + b_2 x)\,(1 + a\,x), \\ &\quad \text{or} \\ &\ddot{x} = b_1 x + b_2 x^2 + a\,(b_1 x + b_2 x^2)\,y, \\ &a \equiv a_1/b_1, \quad b_1 \neq 0. \end{aligned}$$

We repeat all these calculations for the parametrization of $f(x)$ and $g(x)$ functions as polynomial of 6 order and conclude that the integrable cases above correspond to situation with $f(x) = 0$ (at the first) and $f(x) \sim g(x)$ at (the second case).

Scheme

- Calculation of the normal form at the one (or more) stationary points of the system till a some order.
- Calculation finite subsets of equations of **A** condition at these points.
- Solving of these subsets of the equations in system parameters at these stationary points.
- Verifying the fulfillment of the condition **A** for the found parameter sets at these stationary points.
- These parameter sets are used for searching integrals of the system by some methods.

Conclusion

- We represented the Liénard equation as a dynamical system and parameterized it in a polynomial form. For this system, we found two sets of parameters at which it has the first integrals of motion and solvable.
- Both cases are trivial from a point of view of studying of Liénard's equation. The first case corresponds to equality $f(x) = 0$, the second one to $f(x) \sim g(x)$. But the workability of the described technique was illustrated.
- The proposed method for finding suitable integrability parameters has no restrictions on the dimension of the system. It is suitable for any autonomous polynomial system which is solved with respect of the derivations. Restrictions arise only at the stage of searching for integrals of motion.

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Many thanks for your attention