On probability distributions for the boundary states of the Hilbert-Schmidt ensemble of qudits

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Let $\mathcal{H}$ be an $n-$dimensional Hilbert space and unitary group $U(\mathcal{H})$ acts on it preserving the standard Hermitian product. The set $\mathfrak{P}_n$ of density states ϱ of an $n-$dimensional quantum system is distinguished in the cone of non-negatively defined operators on $\mathcal{H}$ by the equation $\mathrm{Tr}(\varrho) = 1$ and can be regarded as embedded in the dual $\mathfrak{u}^*(n)$ of the Lie algebra $\mathfrak{u}(n)$.

It is known that the space $\mathfrak{P}_n$ of quantum states is not a differential manifold with smooth boundary. It is a stratified space, the union of different strata $\mathfrak{P}_n^k$ labelled by the rank $k = 1, 2, \dots n$ of the quantum state, $\dim(\mathfrak{P}_n^k) = 2nk - k^2 - 1$ [1]. However, from the standpoint of dynamics of closed quantum system, it is often important to consider decomposition of the state space $\mathfrak{P}_n$ according to the unitary evolution. In this case a natural decomposition of the state space $\mathfrak{P}_n$ based on the coadjoint action of the unitary group $U(\mathcal{H})$ is relevant. The (co)adjoint action of the group $U(\mathcal{H})$ in $\mathfrak{u}^*(n)$ induces a corresponding non-transitive action on $\mathfrak{P}_n$ and thus different dimension if $k > 1$. Since the interior of the state space $\mathfrak{P}^n$ is a submanifold of the affine subspace of Hermitian operators of a unit trace, non-trivial differential structures in this stratification pattern appear only for the boundary $\partial\mathfrak{P}_n$, consisting of those density states ϱ for which $\det(\varrho) = 0$.

In the present report, we will describe generic features of a geometry of the boundary $\partial\mathfrak{P}_n$, particularly, its Riemannian characteristics in relation with the probability distributions of random states from the Hilbert-Schmidt ensemble of n-dimensional states, i.e., qudits.

An introduction of the notion of a distance between quantum states allows one to endow a quantum state space $\mathfrak{P}$ with a metric structure and thus consider $\mathfrak{P}$ as a Riemannian manifold. This enables us to relate geometrical concepts to a physical ones and use these concepts for studies of statistical properties of quantum systems [2]. The aim of our studies is a derivation of the probability distributions on the boundary $\partial\mathfrak{P}_n$, starting from the flat Hilbert-Schmidt metric on $\mathfrak{u}^*(n)$ [3]. It will be shown that an inherited metric on subsets $\mathfrak{P}_n^k$ gives rise to the joint probability density of a random rank-deficient states $\varrho \in \mathfrak{P}_n^k$, $k < n$, with real eigenvalues $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$ of the so-called β- Wishart-Laguerre ensemble [4, 5]:

$$P_n^{\alpha,\beta}(\lambda_1, \lambda_2, \dots, \lambda_k) = C_{n,\alpha,\beta}|\Delta_k(\{\lambda\})|^\beta \prod_{s=1}^{k} \lambda_s^\alpha e^{-\frac{1}{2}\beta\lambda_s}\,,$$

where $C_{n,\alpha,\beta}$ is a normalization factor, $\Delta_k(\{\lambda\})$ is the Vandermonde determinant. The equation defining parameters α and β as function of the stratum $\mathfrak{P}_n^k$ is derived considering the induced metric on the degenerate unitary orbits $\mathcal{O}_\varrho \in \mathfrak{P}_n^k$.

References

[1] J. Grabowski, G. Marmo and M. Kus, *Geometry of quantum systems: Density states and entanglement*, J. Phys. A **38**, 10217, 2005.

[2] D. Petz, *Quantum Information Theory and Quantum Statistics*, Springer-Verlag, 2008.

[3] I. Bengtsson, K. Zyczkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement*, Cambridge University Press, 2017.

[4] M.L. Mehta, *Random Matrices*, 3rd edn. Academic Press, New York, 2004.

[5] P.J. Forrester, *Log-Gases and Random Matrices*, Princeton University Press, 2010.

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