On Natural Transformations in Compact Closed Categories with Generating Unit

Sergei Soloviev
IRIT, University of Toulouse-3,
118, route de Narbonne, 31062, Toulouse, France,
soloviev@irit.fr

1 What has been done.

The idea that the arbitrary natural transformations of superpositions of distinguished functors (such as tensor product $\otimes : K \times K \to K$ and internal hom-functor $\multimap : K^{op} \times K \to K$) can be described if tensor unit I is a generating object in the category K was first explored in [5] and studied further in [1].

More precisely, the question is formulated as follows. Let K be some category with additional structure including distinguished functors and natural transformations (distinguished objects may be seen as constant functors). Canonical natural transformations are those obtained from distinguished natural transformations by application of functors and composition.[1] Is it possible to describe arbitrary natural transformations between superpositions of distinguished functors in terms of canonical natural transformations with parameters?

The main results below are obtained in the situation when tensor unit I is a generator. One of typical examples where all these results hold is the category of finitely generated projective modules over a commutative ring I with unit.

1.1 SMC and CC Categories

How we may proceed is illustrated by some of the results of [5].

For example, the structure of a symmetric monoidal closed (SMC) category K contains two distinguished functors $\otimes : K \times K \to K$ and $\multimap : K^{op} \times K \to K$. (Tensor product and internal *hom*-functor in typical cases.)

There are also the distinguished object I (tensor unit), the distinguished natural isomorphisms $a_{XYZ} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $b_X : X \otimes I \to X$, $c_{XY} : X \otimes Y \to Y \otimes X$, and the (generalized) natural transformations

[1] Composition of natural transformations and (since the category of functors and natural transformations is 2-category) composition with distinguished functors as well. The latter may be seen as substitution of functorial expressions for variables, e.g., if commutativity of tensor product $c_{XY} : X \otimes Y \to Y \otimes X$ is distinguished natural transformation (thus it is canonical) then $c_{(X\otimes Y)Z} : (X \otimes Y) \otimes Z \to Z \otimes (X \otimes Y)$ is canonical as well.

$d_{XY} : X \to Y \multimap X \otimes Y$, $e_{XY} : (X \multimap Y) \otimes X \to Y$. They must satisfy certain equations. Using these natural transformations, also the *adjunctions* $\pi_{XYZ} : Hom(X \otimes Y, Z) \to Hom(X, Y \multimap Z))$ and its inverse π^{-1} may be defined.[2] (See [4], [5] for detailed definitions.)

One may recall that $I \in Ob(K)$ is a generator iff for any $f \neq g : X \to Y \in Mor(K)$ there exists $h : I \to X$ such that $f \circ h \neq g \circ h$. For example, in the category of sets any non-empty set is a generator. In the category of modules over a ring I, I is a generator.

The key technical lemma in [5] (lemma 2.1) stated that if I is a generator in K then an arbitrary generalized natural transformation $f_X : (X \multimap I) \otimes X \to I$ (in the category of functors over K) is the composition

$$(X \multimap I) \otimes X \overset{e_{XI}}{\to} I \overset{h}{\to} I$$

for some endomorphism $h : I \to I$.

Let F, G be superpositions of distinguished functors represented by formulas in appropriate syntax. Some variables in F, G may be identified; the schema of this identification was called *graph* in [4]. For each occurrence in F, G its variance is defined as usual. The expression $F \to G$ is called the type of a natural transformation $f : F \to G$. For ordinary natural transformations, each variable occurs once in F and once in G (with the same variance). For generalized natural transformations (see, e.g., [4] or [5]) two more cases are admitted: a variable may occur either twice in F or twice in G, with opposite variances (cf. e_{XY} and d_{XY} above). The type $F \to G$ is balanced iff each variable occurs exactly twice (with the same variances when its occurrences lie at the opposite sides of the arrow, and with opposite variances otherwise).

An SMC category K is compact closed (CC) category if there are two more distinguished natural isomorphisms: $s_X : (X \multimap I) \multimap I \to X$ and $t_{XY} : X \multimap Y \to (X \multimap I) \otimes Y$ (their inverses are canonical natural transformations of SMC category, in CC-case they must be isomorphisms).

Let K be compact closed. If the type $F \to G$ is balanced and contains variables $X_1, ..., X_n$ then every generalized natural transformation $g : F \to G$ in K can be obtained from some natural transformation

$$f(X_1...X_n) : ((X_1 \multimap I) \otimes X_1) \otimes ...((X_n \multimap I) \otimes X_n) \to I$$

by composition with canonical natural transformations and applications of adjunctions π and π^{-1}. (For the sake of certainty we assume that $\otimes$, as well as $\oplus$ below, associate to the left.) It may be described as application of some operator Φ depending only on type $F \to G$ to f, i.e., $g = \Phi(f)$. Using lemma 2.1 of [5] we prove that f is equal to the following composition

$$(X_1 \multimap I) \otimes X_1 \otimes ...(X_n \multimap I) \otimes X_n \overset{e_{X_1 I} \otimes ... e_{X_n I}}{\longrightarrow} I \otimes ...I \overset{b_{I \otimes ... I}}{\longrightarrow} ... \overset{b_I}{\to} I \overset{f_0}{\to} I$$

and $g(X_1...X_n) = \Phi(f_0 \circ b_I \circ ... \circ b_{I \otimes ... I} \circ (e_{X_1 I} \otimes ... e_{X_n I}))$. (Cf. Th. 3.10 of [5].)

[2] It is possible also other way round: if π and π^{-1} are considered as basic, e and d may be derived.

Remark 1.1 *Further modifications of this representation are possible. For* $f : X \to Y$*, the* linear multiplication *by an endomorphism* $h : I \to I$ *(denoted* $h \cdot f$*) is defined as the composition*

$$X \xrightarrow{b_X^{-1}} I \otimes X \xrightarrow{h \otimes f} I \otimes Y \xrightarrow{b_Y} Y.$$

Using adjunctions and isomorphisms in a slightly different way, we may show that $g = \Phi'(f)$ *where* $f : X_1 \otimes ... \otimes X_n \to X_1 \otimes ... \otimes X_n$*. Moreover* $f = h \cdot 1_{X_1 \otimes ... \otimes X_n}$*. If we take* $X_1 = ... = X_n = I$*, there exist unique isomorphisms* $\phi : I \to F(I, ..., I), \psi^{-1} : G(I, ..., I) \to I$*, and* $h = \psi^{-1} \circ g(I, ..., I) \circ \phi$*. (We shall note this composition* $g^*(I)$*.)*

It implies also genericity in another sense. When the type $F \to G$ is balanced the equality of two natural transformations $g_1, g_2 : F \to G$ may be checked on I only: $g_1 = g_2 \iff g_1^*(I) = g_2^*(I)$. (Th. 3.12 of [5].)

In [5] the case of CC categories with biproduct $\oplus$ was studied as well, and some results about arbitrary natural transformations $f : F \to G$ where F, G may contain $\oplus$ were proved. The description, however, was partial. The results were subject to some constraints concerning the structure of F, G. For example these results did not cover the case of tensor powers that were not balanced, as $X \otimes X \to X \otimes X$.

1.2 SM Categories with Biproduct

In a more recent work [1] our aim was to obtain full description of natural transformations for arbitrary superpositions of $\otimes$ and $\oplus$ (without constraints such as balancedness).

That is, Symmetric Monoidal (SM) Categories K with $\otimes$ (tensor) and $\oplus$ (biproduct called also direct sum) as distinguished functors were considered. Respectively, there were 2 distinguished objects, tensor unit I and zero-object 0. Distinguished natural transformations for $\otimes$ were the same as before. Distinguished natural transformations for $\oplus$ included canonical projections (for $\oplus$ as product), injections (for $\oplus$ as sum), natural transformations that characterize 0 as zero-object, diagonal and codiagonal maps $\Delta, \bigtriangledown$. Distributivity isomorphisms for $\otimes$ over $\oplus$ in this setting are derived.

Notice that the sum $f + g$ of morphisms $f, g : X \to X$ can be defined as

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} X \oplus X \xrightarrow{\bigtriangledown} X$$

(in fact K is *semi-additive*).

In absence of internal *hom*-functor a slightly modified notion of generation called $tensor - generation$ was necessary. The unit I was called $tensor - generator$ iff given any pair of unequal maps

$$X_1 \otimes ... \otimes X_n \underset{g}{\overset{f}{\rightrightarrows}} X$$

there is a map $k_i : I \to X_i$ such that

$$X_1 \otimes ... \otimes I ... \otimes X_n \xrightarrow{1 \otimes ... k_i \otimes ... 1} X_1 \otimes ... \otimes X_i ... \otimes X_n \overset{f}{\underset{g}{\rightrightarrows}} X$$

are also unequal. That is $1 \otimes ... \otimes k_i ... \otimes 1$ distinguishes f from g. (If internal *hom*-functor $\multimap$ is present as well, I is tensor-generator iff I is generator in ordinary sense because of adjointness of $\otimes$ and $\multimap$.)

Now (proposition 2.3 of [1]) every natural transformation

$$f(X_1, ..., X_n) : X_1 \otimes ... \otimes X_n \to X_{\sigma(1)} \otimes ... \otimes X_{\sigma(n)}$$

may be represented as $f^*(I) \cdot \sigma$ where

$$\sigma : X_1 \otimes ... \otimes X_n \to X_{\sigma(1)} \otimes ... \otimes X_{\sigma(n)}$$

is the canonical natural transformation determined by the permutation σ (i.e. obtained from natural associativity and commutativity of $\otimes$) and $f^*(I)$ is

$$I \xrightarrow{b_I^{-1}} I \otimes I \xrightarrow{b_{I \otimes I}^{-1}} ... I \otimes ... \otimes I \xrightarrow{f(I,...,I)} I \otimes ... \otimes I ... \xrightarrow{b_{I \otimes I}} I \otimes I \xrightarrow{b_I} I.$$

Let F, G be tensor powers (tensor products of variables). The type $F \to G$ was called in [1] multibalanced if the number of occurrences of each variable in F is the same as in G. In K as described above (with biproducts and 0) *each natural transformation* $f : F \to G$ *where* $F \to G$ *is not multibalanced is zero.* (See [1], proposition 4.10.)

Let now $F \to G$ be some type. The type $F' \to G'$ is called its *generalization* if $F \to G$ may be obtained from $F' \to G'$ by identification of some variables. Let $F_1 \to G_1, ..., F_k \to G_k$ be balanced generalizations of a multibalanced type $F \to G$, and $\tau_1, ..., \tau_k$ denote the corresponding identifications of variables (substitutions), i.e., $\tau_i(F_i \to G_i) = F \to G$.

Theorem 1.2 *In the conditions described above, with* $F \to G$ *multibalanced, every natural transformation* $f : F \to G$ *is the sum* $\tau_1(f_1) + ... + \tau_k(f_k)$ *where* $f_1, ..., f_k$ *are some natural transformations of* $F_1 \to G_1, ..., F_k \to F_k$ *respectively. If* $F \to G$ *is not multibalanced then* f *is zero.*

This theorem is a slightly reformulated version of the extraction theorem (theorem 4.2) of [1]. And f_i when I is tensor-generating are of the form

$$F_i \xrightarrow{h_i \cdot \sigma_i} G_i$$

with σ_i canonical natural transformations determined by $F_i \to G_i$ and $h_i : I \to I$ endomorphisms of I. Moreover, $h_i = f_i^*(I)$ (it can be computed using the I-component of f_i).

Example 1.3 *Let $f : X \otimes X \to X \otimes X$. There are two balanced types $X \otimes Y \to Y \otimes X$ and $X \otimes Y \to X \otimes Y$ that produce $X \otimes X \to X \otimes X$ by identification of variables. Then $f = h_1 \cdot c_{XX} + h_2 \cdot 1_{X \otimes X}$. If K is the CC category of finitely dimensional vector spaces, h_1 and h_2 may be seen merely as scalar coefficients.*

Remark 1.4 *Let X^k denote $X \otimes ... \otimes X$ (k times). Up to natural associativity and commutativity each multibalanced type $F \to G$ without $\oplus$ may be seen as $X_1^{k_1} \otimes ... \otimes X_l^{k_l} \to X_1^{k_1} \otimes ... \otimes X_l^{k_l}$. If we "deidentify" the variables of each cluster $X_i^{k_i}$ in a standard way (take at the left $X_{i1} \otimes ... \otimes X_{ik_i}$ and at the right all possible permutations of the same variables) we obtain altogether $n = k_1! \cdot ... \cdot k_l!$ different balanced types that produce $F \to G$ by identification. Then f is equal to the following composition:*

$$F \to X_1^{k_1} \otimes ... \otimes X_l^{k_l} \stackrel{\sum_{i=1}^{n} h_i \cdot \tau(\sigma_i)}{\longrightarrow} X_1^{k_1} \otimes ... \otimes X_l^{k_l} \to G.$$

(Here τ merely identifies back all deidentified variables like in the example above, so it is the same for all i.)

If $\oplus$ occurs in $F \to G$ then by distributivity there exist natural isomorphisms $\phi : F_1 \oplus ... \oplus F_k \to F$ and $\psi^{-1} : G \to G_1 \oplus ... \oplus G_l$ where $F_1, ..., F_k, G_1, ..., G_l$ are tensor products of variables. Thus $f : F \to G$ is $\psi \circ g \circ \phi^{-1}$ for some $g : F_1 \oplus ... \oplus F_k \to G_1 \oplus ... \oplus G_l$. Because $\oplus$ is biproduct, g may be represented by the matrix (g_{ij}) where $g_{ij} = p_j \circ g \circ q_i : F_i \to G_j$ $(1 \leq i \leq k, 1 \leq j \leq l)$, and each natural transformation $g_{ij} : F_i \to G_j$ is described as in theorem 1.2 and remark 1.4.

2 New Advancements

R. Houston proved [3] that if a compact closed category has finite products or finite coproducts then it in fact has finite biproducts, and so is semi-additive. This result shows that in fact CC categories with finite biproducts are much more common than we expected when [5] and [1] were written and incites us to consider more closely what can be obtained if we combine the results of our earlier works.

Let us consider first an arbitrary natural transformation $f : F \to G$ where F, G do not contain $\oplus$. If $F \to G$ is not multibalanced then f is zero. If it is, using composition with canonical natural isomorphisms of the CC-structure and adjunctions π, π^{-1}, f may be represented as $\Psi(f_0)$ where $f_0 : X_1^{k_1} \otimes ... \otimes X_i^{k_i} \to X_1^{k_1} \otimes ... \otimes X_i^{k_i}$. The natural transformation f_0 may be described as in section 1.2, i.e., it is either zero or the sum of $f_0^*(I) \cdot \sigma$.

In CC categories with biproduct not only $\otimes$ distributes over $\oplus$ but also there exists canonical isomorphism $X \oplus Y \multimap I \leftrightarrow (X \multimap I) \oplus (Y \multimap I)$. Using canonical isomorphisms, for any natural transformation $f : F \to G$ one obtains the following commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\quad f \quad} & G \\ \phi \downarrow & & \uparrow \psi^{-1} \\ F_1 \oplus ... \oplus F_k & \xrightarrow{\quad g \quad} & G_1 \oplus ... \oplus G_l \end{array}$$

where vertical arrows represent appropriate canonical isomorphisms and $F_1, ..., F_k$, $G_1, ..., G_l$ do not contain $\oplus$.

The natural transformation g may be represented by the matrix (g_{ij}) where $g_{ij} : F_i \to G_j$. In its turn, g_{ij} is either the sum $\sum_{m=1}^{n_{ij}} \tau_{ij}(g_{ijm})$ if $F_i \to G_j$ is multibalanced or zero otherwise, as in section 1.2. The proof uses the extraction technique of [1] based on properties of biproducts.

The $g_{ijm} : F_{im} \to G_{jm}$ are natural transformations of balanced generalizations of $F_i \to G_j$ obtained by deindentification of variables ($1 \leq m \leq n$, where n is given by factorial expression similar to that considered in remark 1.4). Each g_{ijm} may be described as $h_{ijm} \cdot \sigma_{ijm}$ where $h_{ijm} = g^*_{ijm}(I) : I \to I$ and σ_{ijm} is unique canonical natural transformation of the type $F_{im} \to G_{jm}$ (here the results of [5] are used). The operator τ_{ij} identifies back the variables.

References

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