

On Natural Transformations in Compact Closed Categories with Generating Unit

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What has been done

- The idea that the arbitrary natural transformations of superpositions of distinguished functors
- such as tensor product $\otimes : K \times K \rightarrow K$ and internal hom-functor $\multimap : K^{op} \times K \rightarrow K$
- may be described *at all* is not so evident.
- Notice two kinds of “arbitrary” here:
- arbitrary *superpositions*
- and arbitrary *natural transformations*.
- Of course *naturally* people were interested in arbitrary natural transformations of some “individual” functors and, separately, in, e.g., canonical natural transformations of arbitrary superpositions (such as natural commutativity and associativity).

What has been done

- The idea was first explored in (Soloviev87) and studied further in (CoHySol).
- More precisely, the question is formulated as follows.
- **Let K be some category with additional structure including distinguished functors and natural transformations (distinguished objects may be seen as constant functors).**
- Canonical natural transformations are those obtained from distinguished natural transformations by application of functors and composition.
- **Is it possible to describe arbitrary natural transformations between superpositions of distinguished functors in terms of canonical natural transformations with parameters?**

What has been done

- The main results were obtained in the situation when tensor unit I is a generator.
- The categories with two distinguished functors: tensor product \otimes and internal hom-functor \multimap were considered.
- such as Symmetric Monoidal Closed (SMC) and Compact Closed (CC) Categories.
- One of typical examples is the category of finitely generated projective modules over a commutative ring I with unit.
- How we may proceed is illustrated by some of the results of (Soloviev87).

What has been done

SMC categories

- There are also the distinguished object I (tensor unit),
- the distinguished natural isomorphisms
$$a_{XYZ} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z), b_X : X \otimes I \rightarrow X,$$
$$c_{XY} : X \otimes Y \rightarrow Y \otimes X,$$
- and the (generalized) natural transformations
$$d_{XY} : X \rightarrow Y \multimap X \otimes Y, e_{XY} : (X \multimap Y) \otimes X \rightarrow Y.$$
- They must satisfy certain equations.
- Using these natural transformations, also the *adjunctions*
$$\pi_{XYZ} : \text{Hom}(X \otimes Y, Z) \rightarrow \text{Hom}(X, Y \multimap Z)$$
 and its inverse π^{-1} may be defined.

What has been done

- $I \in \text{Ob}(K)$ is a generator iff for any $f \neq g : X \rightarrow Y \in \text{Mor}(K)$ there exists $h : I \rightarrow X$ such that $f \circ h \neq g \circ h$.
- For example, in the category of sets any non-empty set is a generator. In the category of modules over a ring I , I is a generator.
- Other examples include semi-modules over semi-ring with unit, and
- Pointed sets with “smashed product” as a tensor.

What has been done

- The key technical lemma in (Soloviev87) (lemma 2.1) stated that if I is a generator in K then an arbitrary generalized natural transformation $f_X : (X \multimap I) \otimes X \rightarrow I$ (in the category of functors over K) is the composition

$$(X \multimap I) \otimes X \xrightarrow{e_{XI}} I \xrightarrow{h} I$$

for some endomorphism $h : I \rightarrow I$.

- This permits under certain conditions to describe arbitrary natural transformations of superpositions of \otimes and \multimap .

What has been done

- Let F, G be superpositions of distinguished functors represented by formulas in appropriate syntax.
- Some variables in F, G may be identified; the schema of this identification was called *graph* by Kelly and Mac Lane in (KM). For each occurrence in F, G its variance is defined as usual. The expression $F \rightarrow G$ is called the type of a natural transformation $f : F \rightarrow G$.
- For ordinary natural transformations, each variable occurs once in F and once in G (with the same variance). For generalized natural transformations two more cases are admitted: a variable may occur either twice in F or twice in G , with opposite variances (cf. e_{XY} and d_{XY} above).
- The type $F \rightarrow G$ is balanced iff each variable occurs exactly twice (with the same variances when its occurrences lie at the opposite sides of the arrow, and with opposite variances otherwise).

What has been done

- An SMC category K is compact closed (CC) category if there are two more distinguished natural isomorphisms:
 $s_X : (X \multimap I) \multimap I \rightarrow X$ and $t_{XY} : X \multimap Y \rightarrow (X \multimap I) \otimes Y$ (their inverses are canonical natural transformations of SMC category, in CC-case they must be isomorphisms).
- **Let K be compact closed. If $F \rightarrow G$ is balanced with variables X_1, \dots, X_n then every generalized natural transformation $g : F \rightarrow G$ in K can be obtained from some natural transformation**

$$f(X_1 \dots X_n) : ((X_1 \multimap I) \otimes X_1) \otimes \dots \otimes ((X_n \multimap I) \otimes X_n) \rightarrow I$$

by composition with canonical natural transformations and applications of adjunctions π and π^{-1} .

- For the sake of certainty we assume that \otimes , as well as \oplus below, associate to the left.

What has been done

- The arbitrary natural transformation can be described as application of some operator Φ depending only on type $F \rightarrow G$ to f , i.e., $g = \Phi(f)$.
- By lemma 2.1 of (Soloviev87) f is equal to the following composition

$$(X_1 \multimap I) \otimes X_1 \otimes \dots (X_n \multimap I) \otimes X_n \xrightarrow{e_{X_1 I} \otimes \dots \otimes e_{X_n I}} I \otimes \dots I \xrightarrow{b_{I \otimes \dots I}} \dots \xrightarrow{b_I} I \xrightarrow{f_0} I$$

and

$$g(X_1 \dots X_n) = \Phi(f_0 \circ b_I \circ \dots \circ b_{I \otimes \dots I} \circ (e_{X_1 I} \otimes \dots \otimes e_{X_n I}))$$

Cf. Th. 3.10 of (Soloviev87).

What has been done

- Using adjunctions and isomorphisms, we may show that $g = \Phi'(f)$ where $f : X_1 \otimes \dots \otimes X_n \rightarrow X_1 \otimes \dots \otimes X_n$. Moreover $f = h \cdot 1_{X_1 \otimes \dots \otimes X_n}$.
- **If we take $X_1 = \dots = X_n = I$, there exist unique isomorphisms**

$$\phi : I \rightarrow F(I, \dots, I), \psi^{-1} : G(I, \dots, I) \rightarrow I$$

and

$$h = \psi^{-1} \circ g(I, \dots, I) \circ \phi$$

- **It implies another kind of genericity. When the type $F \rightarrow G$ is balanced, it suffices to check the equality of two natural transformations on I :**

$$g_1 = g_2 \iff g_1^*(I) = g_2^*(I).$$

What has been done

- In (Soloviev87) the case of CC categories with biproduct \oplus was studied as well, and some results about arbitrary natural transformations $f : F \rightarrow G$ where F, G may contain \oplus were proved. The description, however, was partial. The results were subject to some constraints concerning the structure of F, G .
- For example these results did not cover the case of tensor powers that were not balanced, as $X \otimes X \rightarrow X \otimes X$.
- Recently some interesting results about biproducts in CC categories were obtained.
- Our new results are based on combination of these recent works with the results outlined above and with another work on categories with \otimes and \oplus (CoHySol - joint work with R. Cockett and M. Hyland).

SM Categories with Biproduct

- In (CoHySol) our aim was to obtain full description of natural transformations for arbitrary superpositions of \otimes and \oplus (without constraints such as balancedness).
- Symmetric Monoidal (SM) Categories K with \otimes (tensor) and \oplus (biproduct called also direct sum) as distinguished functors were considered.
- Respectively, there were 2 distinguished objects, tensor unit I and zero-object 0 . Distinguished natural transformations for \otimes were the same as before.
- Distinguished natural transformations for \oplus included canonical projections (\oplus as product), injections (\oplus as sum), natural transformations for 0 , diagonal and codiagonal maps Δ, ∇ .
- Distributivity isomorphisms for \otimes over \oplus in this setting are derived.

SM categories with biproduct

- The sum $f + g$ of morphisms $f, g : X \rightarrow X$ can be defined as

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} X \oplus X \xrightarrow{\nabla} X$$

(in fact K is *semi-additive*).

- In absence of internal *hom*-functor a slightly modified notion of generation called *tensor – generation* was necessary. The unit I was called *tensor – generator* iff given any pair of unequal maps

$$X_1 \otimes \dots \otimes X_n \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} X$$

there is a map $k_j : I \rightarrow X_j$ such that

$$X_1 \otimes \dots \otimes I \dots \otimes X_n \xrightarrow{1 \otimes \dots \otimes k_j \otimes \dots \otimes 1} X_1 \otimes \dots \otimes X_j \dots \otimes X_n \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} X$$

are also unequal.

- That is $1 \otimes \dots \otimes k_j \dots \otimes 1$ distinguishes f from g .

SM categories with biproduct

Now (proposition 2.3 of (CoHySol)) every natural transformation

$$f(X_1, \dots, X_n) : X_1 \otimes \dots \otimes X_n \rightarrow X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)}$$

may be represented as $f^*(I) \cdot \sigma$ where

$$\sigma : X_1 \otimes \dots \otimes X_n \rightarrow X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)}$$

is the canonical natural transformation determined by the permutation σ (i.e. obtained from natural associativity and commutativity of \otimes) and $f^*(I)$ is

$$I \xrightarrow{b_I^{-1}} I \otimes I \xrightarrow{b_{I \otimes I}^{-1}} \dots I \otimes \dots \otimes I \xrightarrow{f(I, \dots, I)} I \otimes \dots \otimes I \dots \xrightarrow{b_{I \otimes I}} I \otimes I \xrightarrow{b_I} I.$$

SM categories with biproduct

- Let F, G be tensor powers (tensor products of variables). The type $F \rightarrow G$ was called in (CoHySol) multibalanced if the number of occurrences of each variable in F is the same as in G .
- In K as described above (with biproducts and 0):
- **Each natural transformation $f : F \rightarrow G$ where $F \rightarrow G$ is not multibalanced is zero.**
- (See (CoHySol) , proposition 4.10.)
- The following permits to describe an arbitrary natural transformation as a kind of a matrix.

SM categories with biproducts

- Let now $F \rightarrow G$ be some type. The type $F' \rightarrow G'$ is called its *generalization* if $F \rightarrow G$ may be obtained from $F' \rightarrow G'$ by identification of some variables.
- Let $F_1 \rightarrow G_1, \dots, F_k \rightarrow G_k$ be balanced generalizations of a multibalanced type $F \rightarrow G$, and τ_1, \dots, τ_k denote the corresponding identifications of variables (substitutions), i.e., $\tau_i(F_i \rightarrow G_i) = F \rightarrow G$.

Theorem

In the conditions described above, with $F \rightarrow G$ multibalanced, every natural transformation $f : F \rightarrow G$ is the sum $\tau_1(f_1) + \dots + \tau_k(f_k)$ where f_1, \dots, f_k are some natural transformations of $F_1 \rightarrow G_1, \dots, F_k \rightarrow G_k$ respectively. If $F \rightarrow G$ is not multibalanced then f is zero.

SM categories with biproduct

- This theorem is a slightly reformulated version of the extraction theorem (theorem 4.2) of (CoHySol). And f_i when I is tensor-generating are of the form

$$F_i \xrightarrow{h_i \cdot \sigma_i} G_i$$

with σ_i canonical natural transformations determined by $F_i \rightarrow G_i$ and $h_i : I \rightarrow I$ endomorphisms of I . Moreover, $h_i = f_i^*(I)$ (it can be computed using the I -component of f_i).

Example

Let $f : X \otimes X \rightarrow X \otimes X$. There are two balanced types $X \otimes Y \rightarrow Y \otimes X$ and $X \otimes Y \rightarrow X \otimes Y$ that produce $X \otimes X \rightarrow X \otimes X$ by identification of variables. Then $f = h_1 \cdot c_{XX} + h_2 \cdot 1_{X \otimes X}$. If K is the CC category of finitely dimensional vector spaces, h_1 and h_2 may be seen merely as scalar coefficients.

SM categories with biproduct

Remark. Let X^k denote $X \otimes \dots \otimes X$ (k times). Up to natural associativity and commutativity each multibalanced type $F \rightarrow G$ without \oplus may be seen as $X_1^{k_1} \otimes \dots \otimes X_l^{k_l} \rightarrow X_1^{k_1} \otimes \dots \otimes X_l^{k_l}$. If we “deidentify” the variables of each cluster $X_i^{k_i}$ in a standard way (take at the left $X_{i1} \otimes \dots \otimes X_{ik_i}$ and at the right all possible permutations of the same variables) we obtain altogether $n = k_1! \cdot \dots \cdot k_l!$ different balanced types that produce $F \rightarrow G$ by identification. Then f is equal to the following composition:

$$F \rightarrow X_1^{k_1} \otimes \dots \otimes X_l^{k_l} \xrightarrow{\sum_{i=1}^n h_i \cdot \tau(\sigma_i)} X_1^{k_1} \otimes \dots \otimes X_l^{k_l} \rightarrow G.$$

(Here τ merely identifies back all deidentified variables like in the example above, so it is the same for all i .)

SM categories with biproduct

- If \oplus occurs in $F \rightarrow G$ then by distributivity there exist natural isomorphisms $\phi : F_1 \oplus \dots \oplus F_k \rightarrow F$ and $\psi^{-1} : G \rightarrow G_1 \oplus \dots \oplus G_l$ where $F_1, \dots, F_k, G_1, \dots, G_l$ are tensor products of variables.
- Thus $f : F \rightarrow G$ is $\psi \circ g \circ \phi^{-1}$ for some $g : F_1 \oplus \dots \oplus F_k \rightarrow G_1 \oplus \dots \oplus G_l$.
- **Because \oplus is biproduct, g may be represented by the matrix (g_{ij}) where $g_{ij} = p_j \circ g \circ q_i : F_i \rightarrow G_j$ ($1 \leq i \leq k, 1 \leq j \leq l$), and each natural transformation $g_{ij} : F_i \rightarrow G_j$ is described as above.**

- **R. Houston proved (Houston) that if a compact closed category has finite products or finite coproducts then it has finite biproducts, and so is semi-additive.**
- **This result shows that CC categories with finite biproducts are much more common than we expected when (Soloviev87) and (CoHyS) were written and incites us to consider more closely what can be obtained if we combine the results of our earlier works.**

New advancements

- Let us consider an arbitrary natural transformation $f : F \rightarrow G$ where F, G do not contain \oplus .
- If $F \rightarrow G$ is not multibalanced then f is zero.
- If it is, using composition with canonical natural isomorphisms of the CC-structure and adjunctions π, π^{-1} , f may be represented as $\Psi(f_0)$ where $f_0 : X_1^{k_1} \otimes \dots \otimes X_i^{k_i} \rightarrow X_1^{k_1} \otimes \dots \otimes X_i^{k_i}$.
- The natural transformation f_0 may be described as above, i.e., it is either zero or the sum of $f_0^*(I) \cdot \sigma$.

New advancements

- In CC categories with biproduct not only \otimes distributes over \oplus but also there exists canonical isomorphism $X \oplus Y \multimap I \leftrightarrow (X \multimap I) \oplus (Y \multimap I)$.
- Using canonical isomorphisms, for any natural transformation $f : F \rightarrow G$ one obtains the following commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \phi \downarrow & & \uparrow \psi^{-1} \\ F_1 \oplus \dots \oplus F_k & \xrightarrow{g} & G_1 \oplus \dots \oplus G_l \end{array}$$

where vertical arrows represent appropriate canonical isomorphisms and $F_1, \dots, F_k, G_1, \dots, G_l$ do not contain \oplus .

New advancements

- The natural transformation g may be represented by the matrix (g_{ij}) where $g_{ij} : F_i \rightarrow G_j$. In its turn, g_{ij} is either the sum $\sum_{m=1}^{n_{ij}} \tau_{ij}(g_{ijm})$ if $F_i \rightarrow G_j$ is multibalanced or zero otherwise, as above. The proof uses the extraction technique of (CoHySol) based on properties of biproducts.
- The $g_{ijm} : F_{im} \rightarrow G_{jm}$ are natural transformations of balanced generalizations of $F_i \rightarrow G_j$ obtained by deidentification of variables ($1 \leq m \leq n$, where n is given by factorial expression similar to that considered in the remark above).
- Each g_{ijm} may be described as $h_{ijm} \cdot \sigma_{ijm}$ where $h_{ijm} = g_{ijm}^*(I) : I \rightarrow I$ and σ_{ijm} is unique canonical natural transformation of the type $F_{im} \rightarrow G_{jm}$ (here the results of (Soloviev87) are used).
- The operator τ_{ij} identifies back the variables.

THANKS FOR YOUR ATTENTION!

References

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